



PB99-100323

REPORT FHWA/NY/SR-92/108

Analysis of Freeway Traffic Speed By Power Spectral Density Methods

JIAN LU



SPECIAL REPORT 108

**ENGINEERING RESEARCH AND DEVELOPMENT BUREAU
NEW YORK STATE DEPARTMENT OF TRANSPORTATION**
Mario M. Cuomo, Governor / Franklin E. White, Commissioner

REPRODUCED BY: **NTIS**
U.S. Department of Commerce
National Technical Information Service
Springfield, Virginia 22161

ANALYSIS OF FREEWAY TRAFFIC SPEED
BY POWER SPECTRAL DENSITY METHODS

Jian Lu, Engineering Research Specialist I

Special Report 108
October 1992

PROTECTED UNDER INTERNATIONAL COPYRIGHT
ALL RIGHTS RESERVED.
NATIONAL TECHNICAL INFORMATION SERVICE
U.S. DEPARTMENT OF COMMERCE

Reproduced from
best available copy.



Engineering Research and Development Bureau
New York State Department of Transportation
State Campus, Albany, New York 12232

ABSTRACT

This report presents results of a study using a Maximum Entropy Spectral Analysis (MESA) approach to evaluate driver behavior as related to driving speed taking into consideration heavy and light traffic conditions, rainy and dry weather conditions, impacts of traffic entering from ramps, ramp impacts, and operating characteristics of various drivers. Three test sites were chosen in the Albany area. Testing vehicle speed between two fixed points was recorded and transferred to spectral density functions. Impacts of traffic conditions, weather conditions, vehicles entering from ramps, and driver behavior can be identified from these spectral density characteristics. Basic concepts of maximum entropy spectral analysis and results of the field experiments are discussed and presented in the Appendix.

CONTENTS

I. INTRODUCTION	1
II. THE MESA CONCEPT	3
A. Discrete Spectral Transformation	3
B. MESA of Vehicle Speed Characteristics	5
III. CONSIDERATIONS IN THE FIELD EXPERIMENTS	7
IV. FIELD EXPERIMENTS	11
A. Spectral Characteristics of Driver Behavior Under Varied Traffic Conditions	11
B. Weather Condition Impact on Vehicle Speed	11
C. Ramp Impact on Vehicle Speed	17
D. Driver Behavior Identification	17
V. DISCUSSION	19
ACKNOWLEDGMENTS	21
REFERENCES	23
APPENDIX: The Concept of Maximum Entropy Spectral Estimation	

Figure 1. Point and section detection of traffic flow.

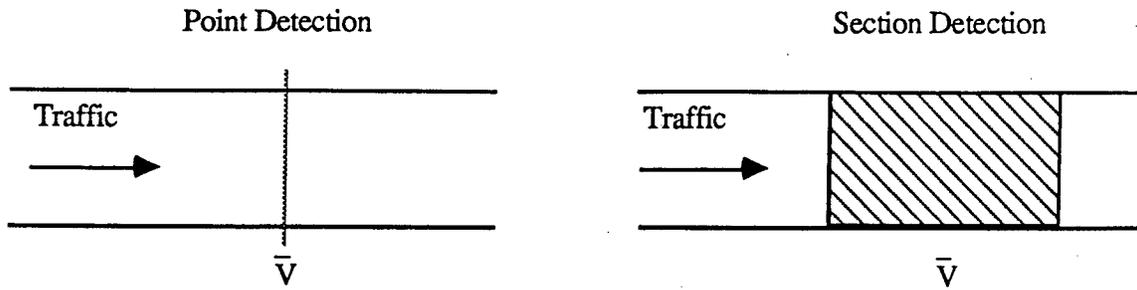
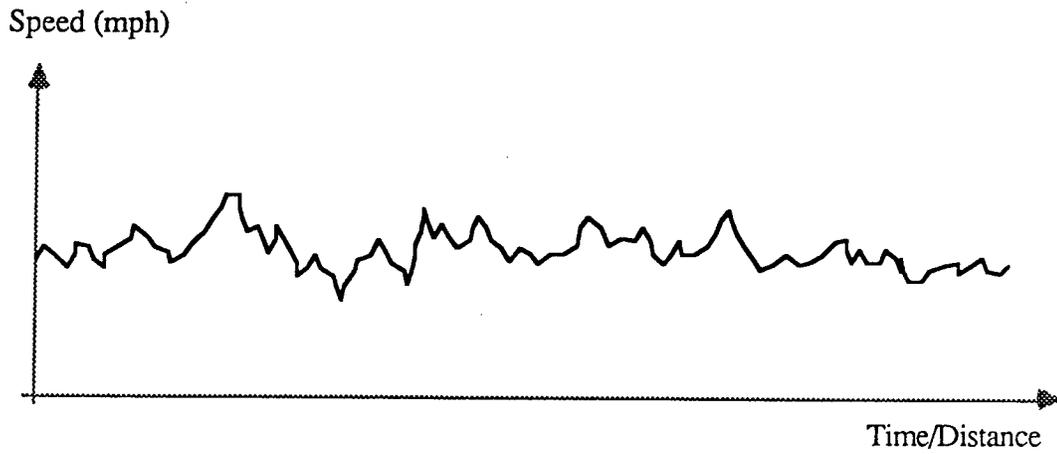
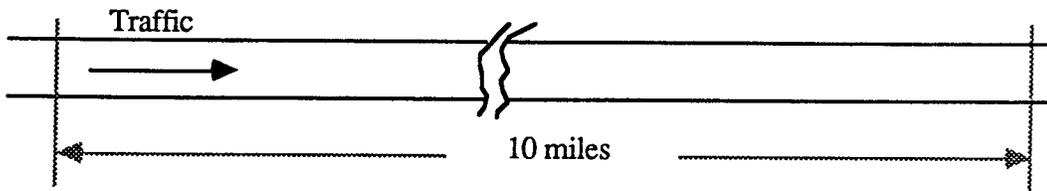


Figure 2. Line detection of traffic flow.



I. INTRODUCTION

Currently, from the perspective of traffic control and traffic flow analysis, local or section speed is the main traffic flow characteristic to be monitored and fed back to a control system. Conceptually, local or section speed detection may be considered a "point-detection" technique -- i.e., calculated data statistically represent traffic characteristics at a specific highway location or within a relatively short section, without considering the dynamic process of a moving vehicle on a relatively long highway section. Basic concepts for point and section detection are shown in Figure 1.

The dynamic process of a moving vehicle on a highway section, say 10-miles long, is affected by traffic conditions, environmental conditions, geometric conditions, driver behavior, etc., within that section. Detection of a dynamic process can be considered "line-detection," and can provide sufficient and more accurate information for traffic control than "point detection." Figure 2 shows the concept of line detection. The data actually resulting from line detection represent the relation of speed history vs. time or distance of the testing vehicle. This speed history reflects speed-control characteristics of individual drivers, which thus are microscopic rather than macroscopic. However, dynamic speed processes controlled by different drivers or impacted by different outside conditions may not be clearly identified in the time or space domains. The research scope for microscopic characteristics of traffic flow is sometimes limited because of lack of methodology or limitations of those domains.

The Engineering Research and Development Bureau recently completed a research study to evaluate traffic flow along a line in the frequency domain, rather than the time or space domains. The spectral analysis method was used, which correspondingly provides a new analysis scope for traffic flow evaluation.

Spectral analysis techniques have been used in transportation engineering for many years, in such areas as pavement surface roughness (1), traffic flow prediction (2), and spacing of transverse pavement cracks (3). The study reported here evaluated driver behavior related to driving speed as affected by heavy and light traffic, rainy and dry weather, and entering traffic from ramps, using a Maximum Entropy Spectral Analysis (MESA) approach. The basic idea of MESA is to transfer individual vehicle speed recorded in the time domain to spectral density in the frequency domain by the maximum entropy spectral estimate method (MESE) (4). Impacts of traffic conditions, weather conditions, vehicles entering from ramps, and driver behavior can be identified from spectral density characteristics. Because the impact of traffic lights was not a study variable, interstates I-87, I-90, and I-787 in Albany were chosen as test sites.

It should be emphasized here that this study's purpose was to provide a new method for traffic flow analysis or highway operations. Application of MESA was

this study's main objective. Theoretical derivation and discussion will not be emphasized. The study reported reflects only initial research results showing how the models work.

II. THE MESA CONCEPT

A. Discrete Spectral Transformation

Vehicle speed data sampled at time interval T can be abstracted as a discrete sequence, called the "discrete speed sequence" or

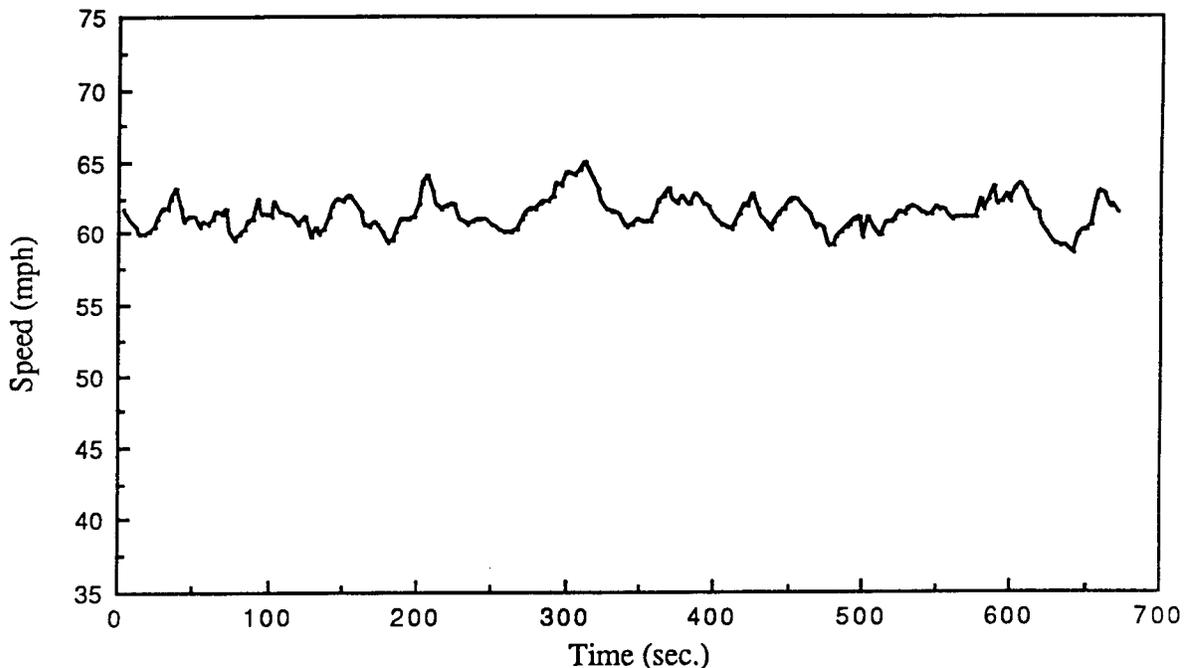
$$\{V_i\} = \{V_1, V_2, \dots, V_N\} \quad (1)$$

where V_i = i^{th} vehicle speed data sampled at time interval T (sec). Figure 3 shows a typical discrete speed sequence collected from I-87 during a non-rush-hour period. Consider the inverse discrete Fourier transformation of the discrete speed sequence defined by Oppenheim (5):

$$V_i = \frac{1}{N} \sum_{k=0}^{N-1} H_k e^{j i k 2\pi / N} \quad (i = 1, 2, \dots, N) \quad (2)$$

where N = length of the sequence (number of data points in the sequence)

Figure 3. Speed sequence collected from Site 1 (light traffic, dry, all lanes).



$$\begin{aligned} V_i &= i^{\text{th}} \text{ vehicle speed data} \\ H_k &= \text{weights } (k = 0, 1, 2, \dots, N - 1), \text{ and} \\ j &= (-1)^k. \end{aligned}$$

Equation 2 states that V_i can be considered a weighted summation of sine functions $e^{jik2\pi/N}$. If a new variable ω_k is defined by

$$\omega_k = K2\pi/N \quad (K = 0, 1, \dots, N-1)$$

then

$$V_i = \frac{1}{N} \sum_{K=0}^{N-1} H_k e^{j\omega_k i} \quad (3)$$

and

$$e^{j\omega_k i} = \sin \omega_k i + j \cos \omega_k i$$

Usually, the variable ω_k is called frequency and is within the range of $0, 2\pi(N-1)/N$. From Equation 2, it is known that the larger the H_k , the more sine function components with frequency ω_k are contained in the discrete speed sequence $\{V_i\}$. Mathematically, it can be proved that

$$H_k = H(\omega_k) = \sum_{i=-\infty}^{+\infty} V_i e^{-j\omega_k i} \quad (\omega_k = 0, 2\pi/N, 4\pi/N, \dots, 2\pi(N-1)/N) \quad (4)$$

In other words, $H(\omega_k)$ is the discrete Fourier transformation of $\{V_i\}$ and the function of frequency ω_k . Equation 4 implies that the discrete speed sequence $\{V_i\}$ in the space domain can be transferred into the frequency domain sequence $\{H(\omega_k)\}$, and characteristics of sequence $\{V_i\}$ can be analyzed in the frequency domain -- i.e., knowing $H(\omega_k)$, one can analyze the characteristics of $\{V_i\}$. Since $H(\omega_k)$ is an imaginary sequence, a real function is defined by

$$S(\omega_k) = |H(\omega_k)|^2 \quad (5)$$

where $S(\omega_k)$ is called the spectral density function of sequence $\{V_i\}$. To calculate $H(\omega_k)$ from Equation 4, the summation should be from $-\infty$ to $+\infty$. In practical engineering cases, sequence length N is finite because one cannot collect infinite numbers of data. The spectral density function $S(\omega_k)$ thus should be estimated from $\{V_i\}$ by some estimation model, instead of using Equation 4.

In the area of spectral function estimation, several mathematical methods are available, such as fast Fourier transformation (FFT) (5), maximum-likelihood spectral estimation (4), and maximum-entropy spectral estimation (4). FFT is one of the most popular spectral estimation models, but it is often difficult to analyze signals by FFT because of their discrete and random characteristics. On the other hand, the "window" requested by FFT calculation usually results in significant spectral estimation error. The maximum entropy spectral estimation (MESE) method is a non-linear and parametric estimation model that does not have "window" and "discrete" problems.

B. MESA of Vehicle Speed Characteristics

From this discussion it can be understood that the spectral density function $S(\omega_k)$ represents the frequency density distribution characteristics of the discrete speed sequence $\{V_i\}$. The basic idea is that if the discrete speed sequence $\{V_i\}$ changes smoothly, or the driver drives his vehicle in a steady manner, then $S(\omega_k)$ contains relatively numerous low-frequency components. This means that the magnitude of $S(\omega_k)$ in the high-frequency region is fairly low. On the other hand, if the discrete speed sequence changes randomly, or the driver changes his speed abruptly, then $S(\omega_k)$ contains relatively numerous high-frequency components, and the magnitude of $S(\omega_k)$ in the high-frequency region thus is relatively higher.

Conceptually, the spectral density function in the low-frequency region represents contour characteristics or macroscopic characteristics of a speed curve in a long period, but the spectral density function in the high-frequency region represents detail changes or microscopic characteristics of the speed curve in a short period. Since frequency of speed change is limited, spectral density characteristics should be band-limited. Figure 4 shows the spectral density function of the speed curve presented in Figure 3. From this graph, it is known that the spectral density function is band-limited, and low-frequency components dominate the whole spectral density function. In fact, the spectral density function shown in Figure 4 is a typical model of speed spectral density characteristics.

Figure 4. Spectral density function of speed sequence (Site 1, light traffic, dry, all lanes).

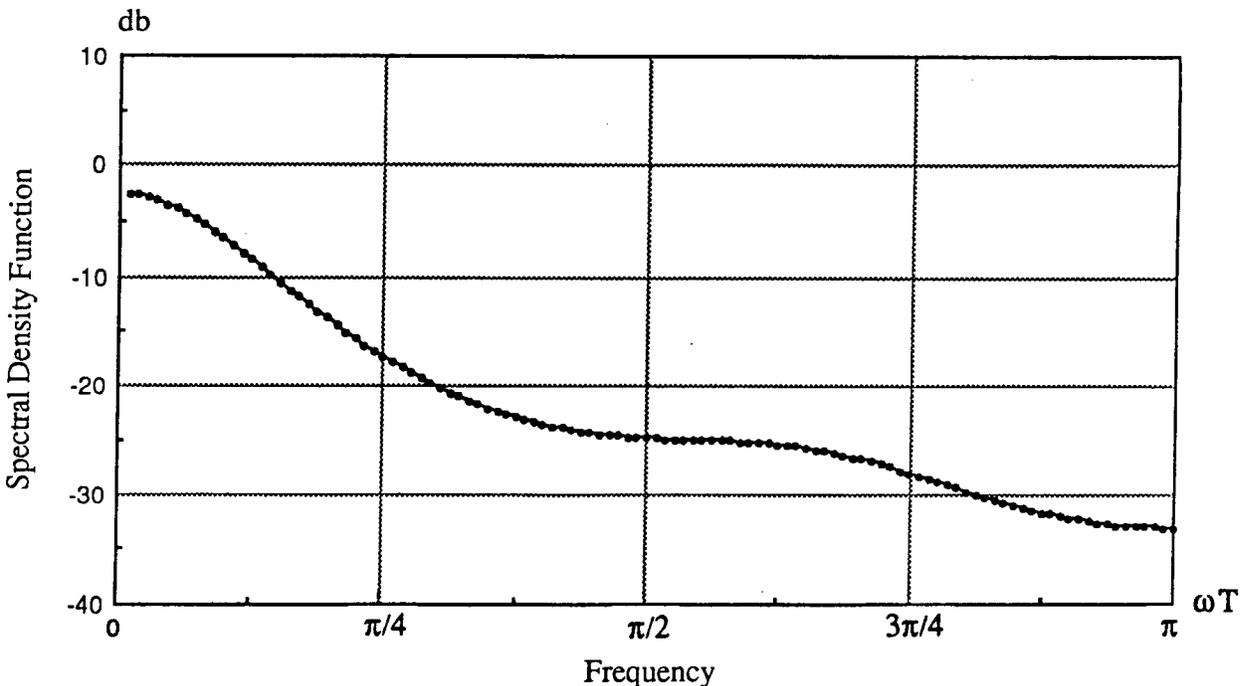
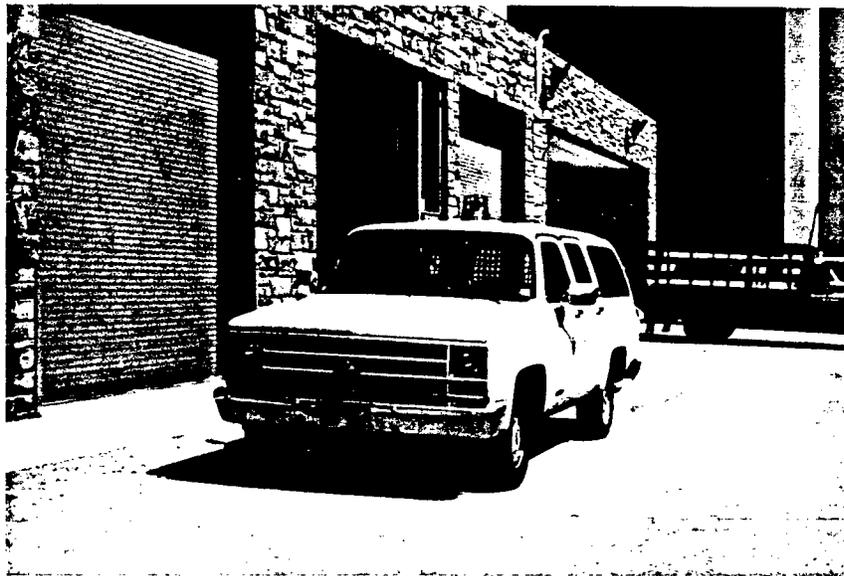


Figure 5. Data sampling equipment and testing vehicle.



III. CONSIDERATIONS IN THE FIELD EXPERIMENTS

Field experiments were conducted to check if maximum entropy spectral density characteristics can identify dynamic speed characteristics under various conditions. The following conditions were considered in the field experiments:

1. Heavy and Light Traffic

"Heavy" or "light" traffic conditions refer to rush-hours or non-rush-hours, respectively. In the Albany area, morning rush-hour usually happens between 7 and 8 a.m. In this study, heavy traffic speed data were collected at the morning rush-hour and light traffic speed data were collected between 10 and 11 a.m.

2. Weather Condition

In this study, weather conditions are described as rainy or dry, and results are based on non-rush-hour or light traffic flow. Generally, for the rainy condition, pavement should be wet enough so that pavement skid resistance is significantly different from the dry condition.

3. Ramp Impact

Traffic entering from a ramp affects moving traffic in a freeway. Compared with the left lane, the right lane should be more affected by entering traffic. This study used two cases to evaluate ramp impact. In the first, the testing vehicle was required to move only in right lane, undergoing more ramp impact; in the second, it was allowed to move in any lane. Ramp impact can be found from these two cases. Tests were based on the non-rush-hour traffic condition.

4. Driver Behavior Comparison

Speed characteristics controlled by different drivers are one of the concerns in this study. Two drivers were selected to run the testing vehicle in heavy and light traffic conditions, and their speed spectral characteristics were computed to identify their differences in driving behavior.

Speed data were collected by a speed transducer measuring transmission angle speed and an instrument called the Fluke Meter. The speed transducer was well calibrated to a radar detector. Figure 5 shows the Fluke Meter and testing vehicle. The process of sampling field data is relatively simple: the testing vehicle was driven from Site A to Site B and its speed was sampled at 3-sec intervals by the Fluke Meter. Then recorded data including speed, traffic condition, test site identification, weather condition, lane change, driver's

Figure 6. Albany field experimental locations.

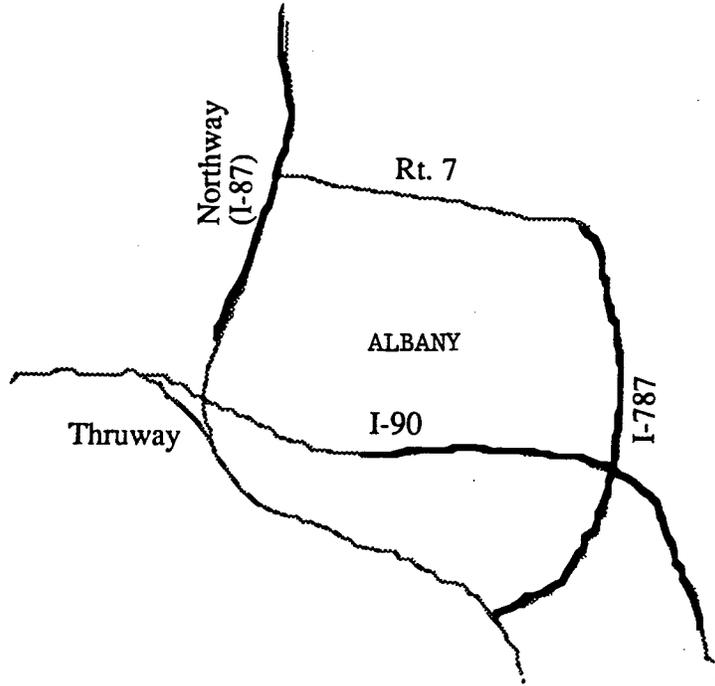


Figure 7. Factorial for field experiments.

		Rainy		Dry	
		Heavy Traffic	Light Traffic	Heavy Traffic	Light Traffic
Driver A	Right Lane				
	All Lanes				
Driver B	Right Lane				
	All Lanes				

name, date, and other information were sent to lab for data reduction and analysis.

Basic field test requirements were as follows:

1. Test Site Length

This should be long enough so the basic dynamic process of changing speed can be recorded. In this study, test site length was about 10 miles.

2. Traffic Flow Condition

To find the difference of spectral density characteristics under heavy and light traffic-flow conditions, test sites should have heavy flow during rush-hour periods and light flow during non-rush-hour periods.

3. Freeway Exits

A few ramps should be included because ramp impacts were to be considered.

Combining these requirements, three test sites in the Albany area were chosen at the locations are shown in Figure 6:

1. I-87 between Exits 2 and 9 (southbound)
2. I-90 between Exits 5 and 10 (eastbound)
3. I-787 between Route 9W and Tibbets Ave. (southbound).

During testing, driver behavior should be as objective as possible -- i.e., his speed control characteristics should change according to traffic conditions, ignoring the fact that he is in a test situation. Figure 7 shows the field test factorial in which "Right Lane" means the testing vehicle always stays to the right (to compare ramp impacts), and "All Lane" means the driver can change lane depending on traffic conditions. In an "All lane" case, impact of the ramp is less than in "Right Lane." It should be noted that this so-called "factorial" does not mean the factorial design in statistical analysis, but refers to the field experiments evaluating the variables listed in Figure 7.

IV. FIELD EXPERIMENTS

A. Spectral Characteristics Of Driver Behavior Under Varied Traffic Conditions

Heavy and light traffic conditions are two extremely different cases, in which a driver may control his vehicle speed differently. Generally, when traffic is light, vehicle speed is relatively stable as compared with heavy traffic conditions. However, this difference may not be easy to identify in the time or space domains. Figures 8, 9, and 10 show speed data collected from Sites 1, 2, and 3, respectively, representing light traffic during non-rush-hour periods, and also heavy traffic during rush-hours. Spectral density characteristics of these speed data are presented in Figures 11, 12, and 13 (vertical scale: $20\log |S(\omega_k)|$ was used), from which it can be seen that these characteristics differ significantly under heavy and light traffic volumes, although these differences cannot be easily identified from Figures 8, 9, and 10. Magnitude of the spectral density function under heavy traffic is much greater than under light traffic, which means (as stated earlier) that the driver may change his speed abruptly because of heavy traffic ahead of his vehicle. Statistically, magnitude of the spectral density function resulting from heavy traffic is higher than that from light traffic. Figure 14 shows speed curves collected from Site 1 under very heavy traffic. Figure 15 shows spectral density functions resulting from speed data shown in Figure 3 representing light traffic, from Figure 8 representing heavy traffic, and from Figure 14 representing very heavy traffic. It is known that the heavier the traffic, the higher is the magnitude of the spectral density function.

B. Weather Condition Impact on Vehicle Speed

In this study, weather condition is described as rainy or dry, and results are based on non-rush-hour traffic flow. Since no heavy rain occurred during testing, it cannot be discussed here. A major concern was whether rain has significant impact on individual vehicle speed by spectral analysis. The literature indicates that a wet pavement surface has less skid resistance, which affects driving safety characteristics. But it should be known whether a wet pavement surface significantly affects driver behavior in terms of speed. In this study, a few tests were conducted at the three sites to study rainy weather impact. It would be expected that during rain, a driver might keep cautiously adjusting his speed to find a desired level that he considers safe. The driver might not accelerate or decelerate if rain is not heavy. Thus, the shape of spectral density characteristics could be used in analyzing rainy weather impact. A suitable way to identify curve shape is use of normalized spectral density curves. Figures 16, 17, and 18 show normalized spectral density characteristics under rainy and dry conditions, with speed data collected from Sites 1, 2, and 3. From these graphs, it is apparent that spectral density characteristics of vehicle speed under rainy and dry weather do not differ significantly -- i.e.,

Figure 8. Site 1 speed sequence (dry, all lanes).

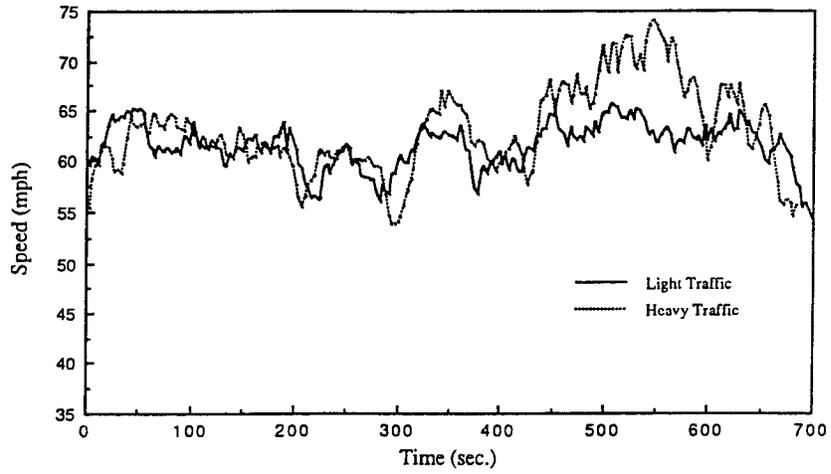


Figure 9. Site 2 speed sequence (dry, all lanes).

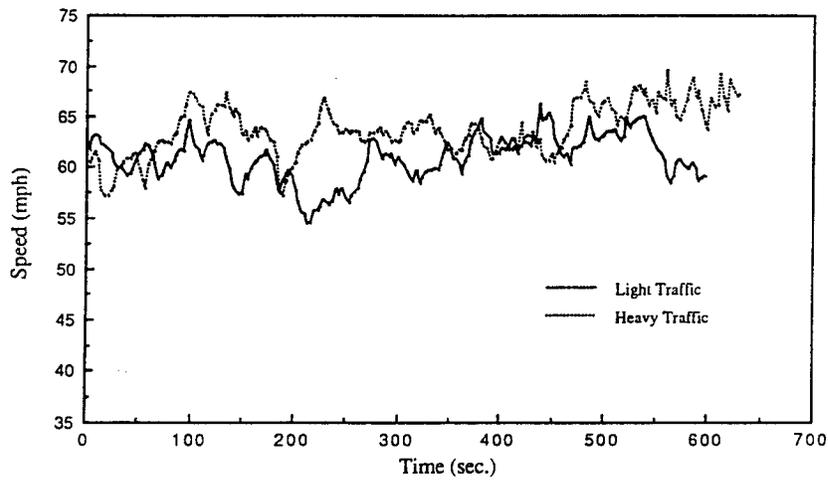


Figure 10. Site 3 speed sequence (dry, all lanes).

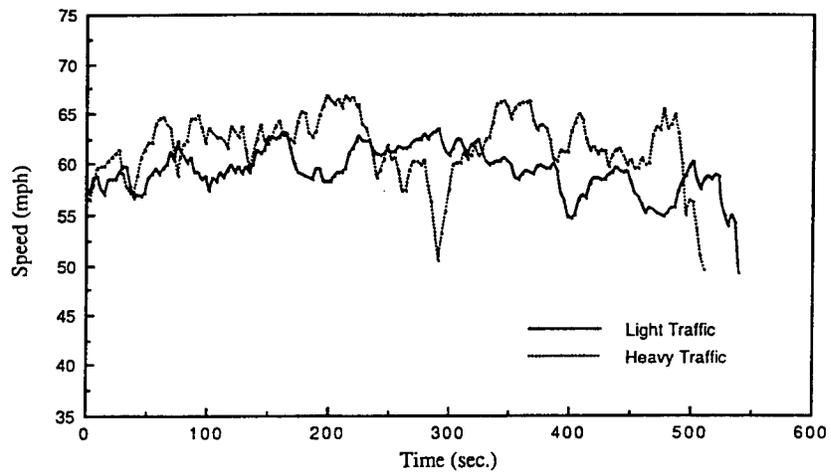


Figure 11. Site 1 spectral density functions of speed sequence under light and heavy traffic (dry, all lanes).

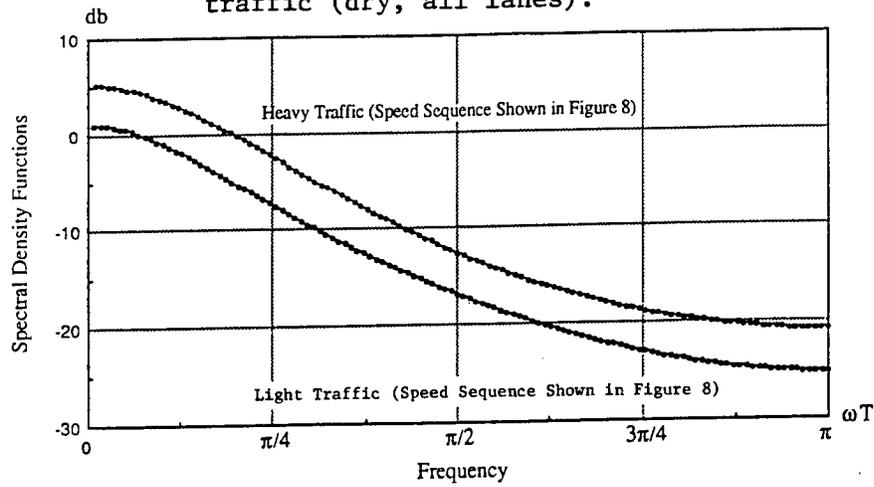


Figure 12. Site 2 spectral density functions of speed sequence under light and heavy traffic (dry, all lanes).

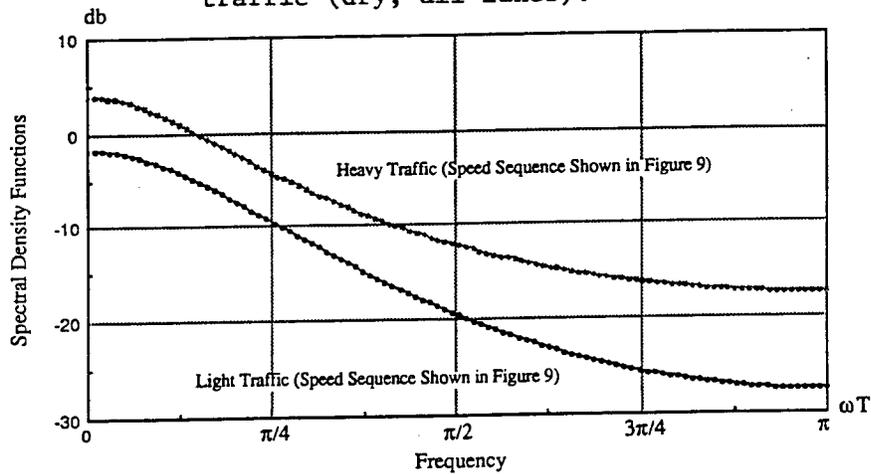


Figure 13. Site 3 spectral density functions of speed sequence under light and heavy traffic (dry, all lanes).

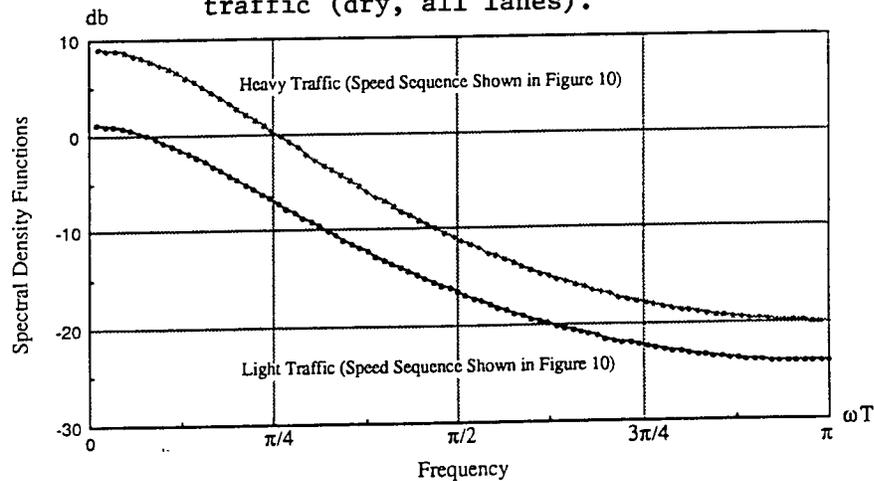


Figure 14. Site 1 speed sequence (very heavy traffic, dry, all lanes).

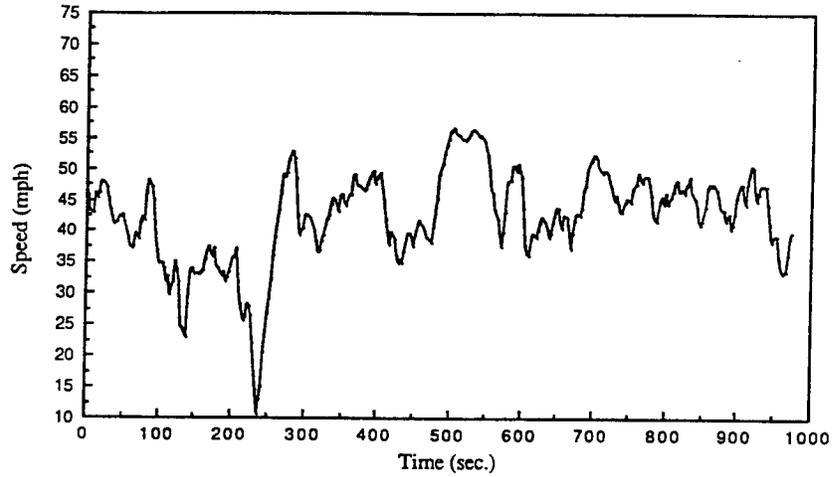


Figure 15. Site 1 spectral density functions of speed sequences under three traffic conditions (dry, all lanes).

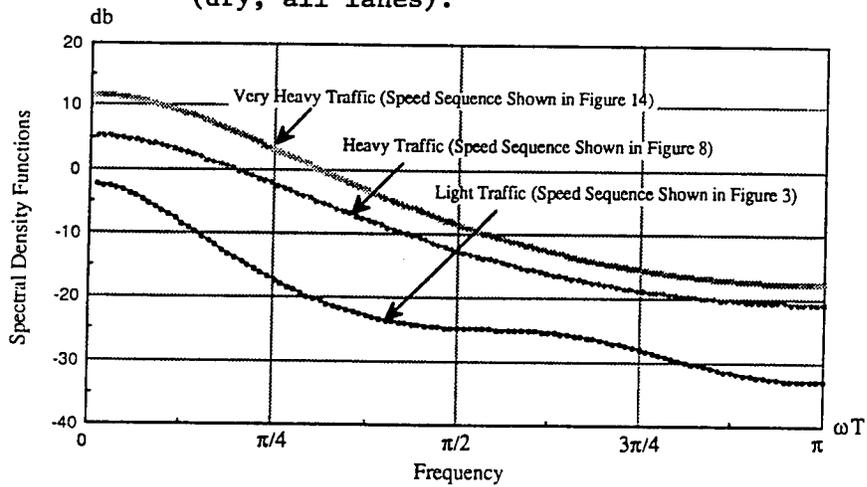


Figure 16. Site 1 normalized spectral density functions in two weather conditions (light traffic, all lanes).

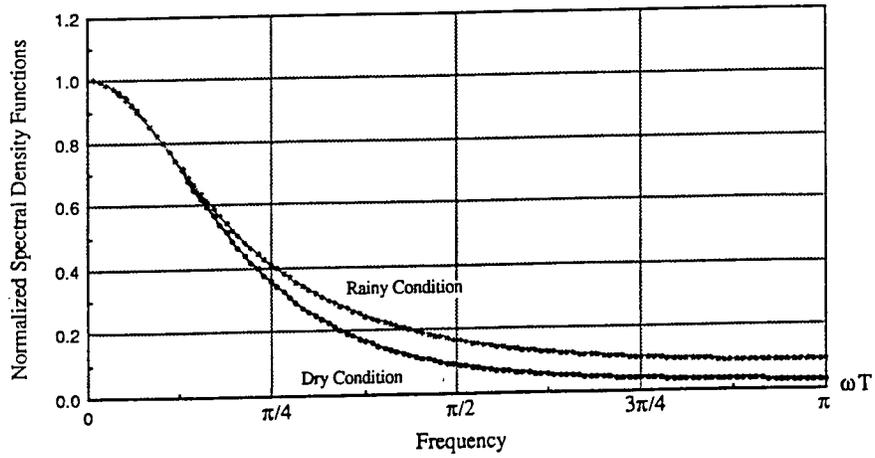


Figure 17. Site 2 normalized spectral density functions in two weather conditions (light traffic, all lanes).

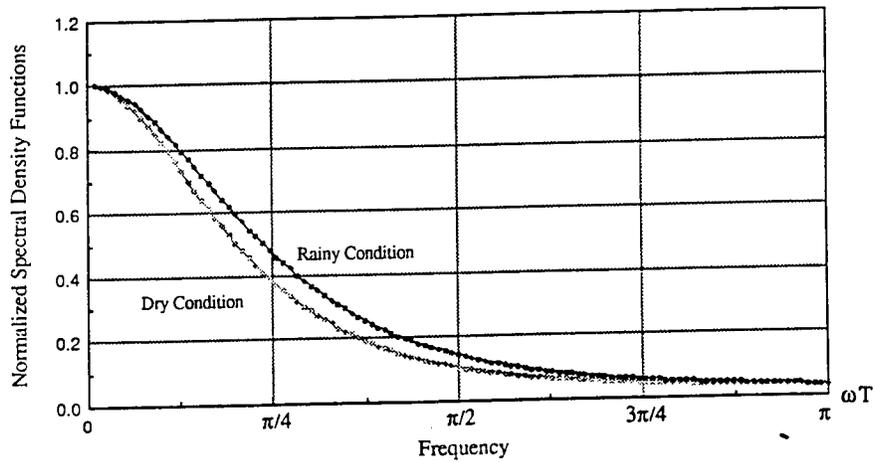


Figure 18. Site 3 normalized spectral density functions in two weather conditions (light traffic, all lanes).

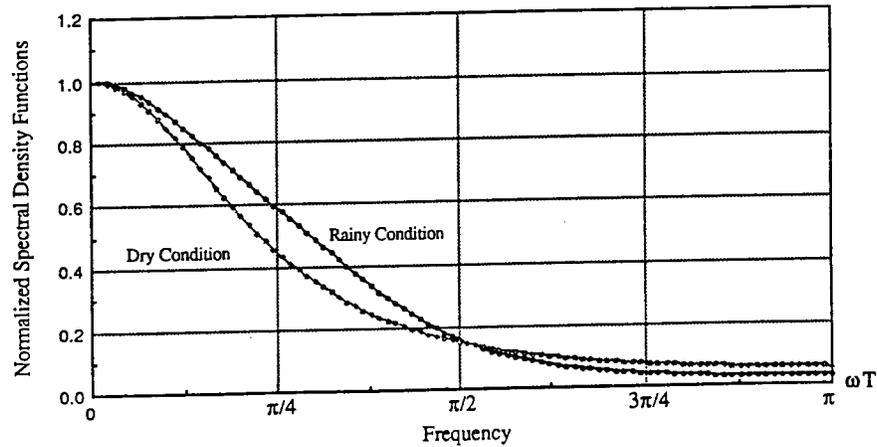


Figure 19. Site 1 normalized spectral density functions in right-lane and all-lane conditions (light traffic, dry).

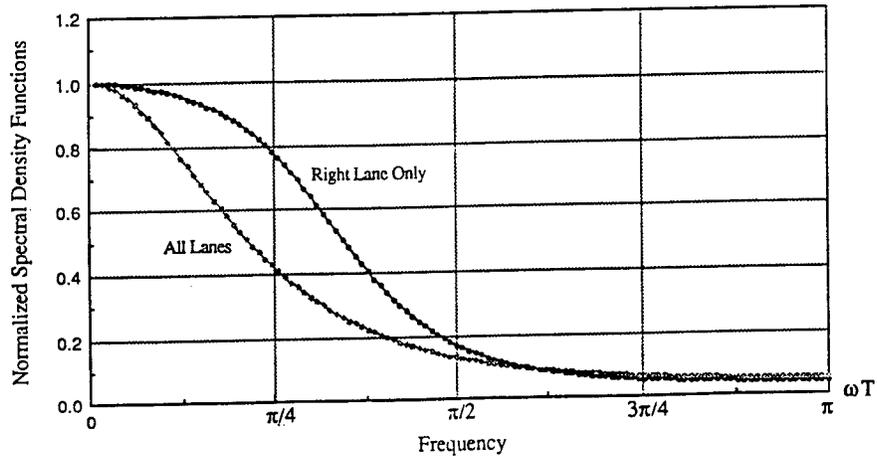


Figure 20. Site 2 normalized spectral density functions in right-lane and all-lane conditions (light traffic, dry).

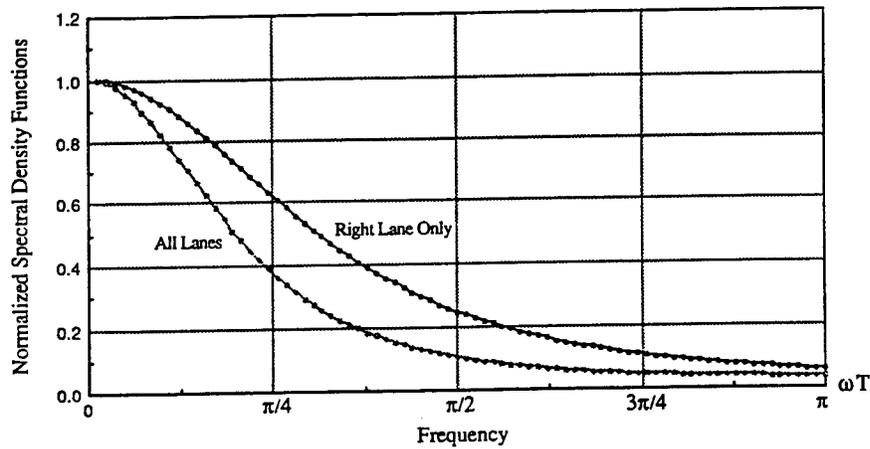
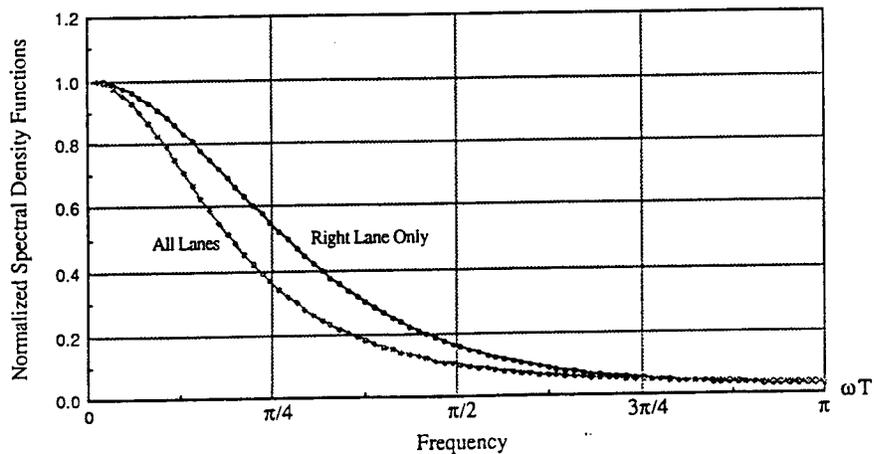


Figure 21. Site 3 normalized spectral density functions in right-lane and all-lane conditions (light traffic, dry).



driver behavior in terms of speed is not significantly affected by wet pavements. However, during field testing, no heavy rain occurred, and the results may not be applicable to such conditions.

C. Ramp Impact on Vehicle Speed

Vehicles entering from a ramp significantly affect speed characteristics of vehicles already in a freeway. In recent years, research has been done on macroscopic characteristics of ramp impact in time or space domains. One objective of this study was to evaluate vehicle-speed spectral density characteristics affected by vehicles entering from a ramp during a heavy-traffic condition. It was assumed here that vehicles staying in the right lane were more affected by entering vehicles than those that could change lanes when approaching ramps. Two cases were considered. First, the test vehicle was directed to stay in the right lane no matter how bad traffic was. Second, the testing vehicle was allowed to change lanes to avoid ramp impact. When vehicle speed approached zero the test was considered "fail." Tests were conducted at Sites 1, 2, and 3, and corresponding normalized spectral density functions are shown in Figures 19, 20, and 21, from which differences of "right lane" and "all lane" can be identified.

D. Identification of Driver Behavior

The tests discussed so far were based on speed characteristics controlled by a specified driver called "Driver A." However, drivers behave differently -- i.e. some control vehicles in an aggressive or unstable manner and others defensively or stably. In the frequency domain, difference of driving behavior can be spotted. Conceptually, an aggressive driver adjusts his speed more often and more quickly under various traffic conditions than a defensive driver does, resulting in higher spectral density magnitude in the whole frequency range. In this study, field tests were conducted at Site 1 to examine this assumption. Two drivers (designated A and B) were selected from the research staff, and traffic condition (heavy or light) was considered. Figures 22 and 23 show differences of Drivers A and B spectral density characteristics under heavy and light traffic conditions. From these graphs, Driver A had a higher spectral magnitude than Driver B, meaning that he controlled his vehicle more aggressively, but this difference was relatively less when traffic was light than when traffic was heavy.

Figure 22. Light traffic spectral density functions of speed sequences for two drivers (Site 1, dry, all lanes).

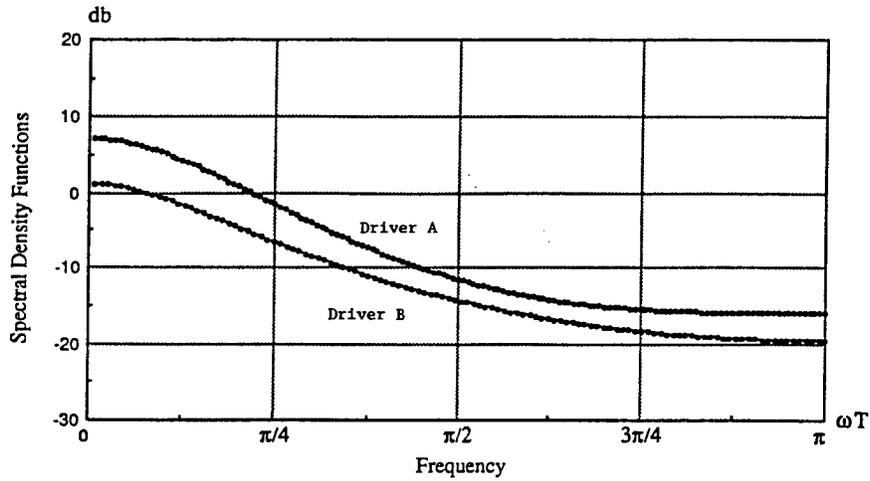
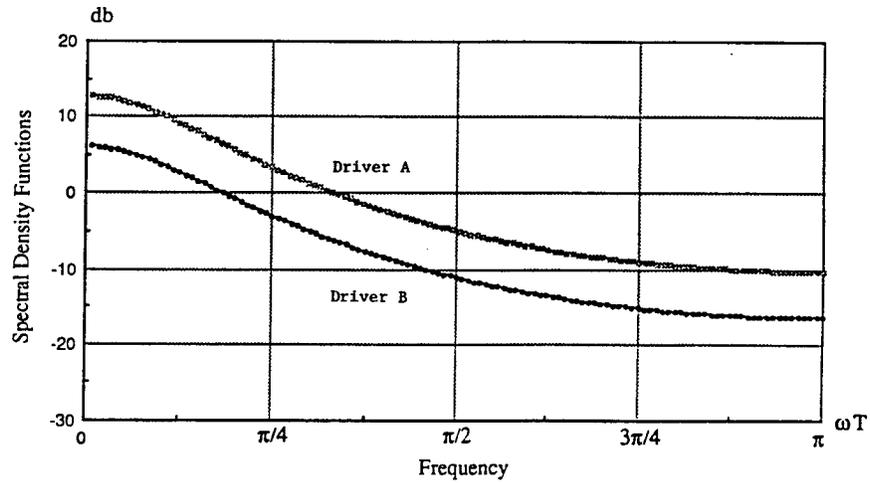


Figure 23. Heavy traffic spectral density functions of speed sequences for two drivers (Site 1, dry, all lanes).



V. DISCUSSION

1. The spectral analysis technique has been accepted in many engineering areas, but not widely applied nor evaluated in transportation engineering. In fact, in addition to time and space domains, spectral analysis provides another analytical alternative. Some problems that cannot be solved in those domains may be solved easily in the frequency domain.
2. Individual vehicle speed is a stochastic process. If data sampling is limited to a certain time period, this process can be assumed to be symptomatically stationary and approximately normally distributed without obvious constant trends. This assumption makes non-linear spectral estimation methods applicable.
3. The study reported reflects only initial research results showing how the models work. Actual application might require considerable experience. More effort is needed to evaluate these traffic impacts.
4. Most current vehicle-speed-related research is based on "point detection" or "section detection," i.e., the mean value of measured speeds is taken as the main variable. In this way more important information is "averaged." In fact, such information can be obtained from detection of the dynamic speed process, which is called "line detection." The technique discussed in this report belongs to line detection.
5. Data collected from line detection can be analyzed in the time/space domains. Analysis in the frequency domain has been widely used in continuous and discrete control systems and signal evaluation. The spectral analysis technique discussed here can be used to assess highway level-of-service (LOS), traffic congestion, and safety of the traveling public. This technique can also be used in detecting traffic incidents and traffic control, such as Intelligent Vehicle Highway Systems (IVHS). However, further research is needed to apply spectral analysis techniques to these areas.
6. Maximum entropy spectral estimate (MESE) is one of the spectral estimate methods. For spectral analysis, other estimate methods could be used. Many computer software packages are available in the market.

ACKNOWLEDGMENTS

William G. Roth, Laboratory Equipment Designer, performed instrumentation and data reduction work for this study. Field tests were conducted by Robert W. Rider, Civil Engineer I, Andrew A. Dagostino, Principal Engineering Technician, and Edward W. Bikowitz, Senior Engineering Technician.

REFERENCES

1. B. G. Hutchinson. Analysis of Road Roughness Records by Power Spectral Density Techniques. Report RR 101, Ontario Department of Highways, January 1965.
2. H. Nicholson and C. D. Swann. "The Prediction of Traffic Flow Volumes Based on Spectral Analysis." Transportation Research, Vol. 8, 1974, pp. 533-538.
3. J. Lu, B. F. McCullough, and C. L. Saraf. "Maximum Entropy Spectral Analysis of Transverse Crack Spacing in Continuously Reinforced Concrete Pavements." Transportation Research Record 1227, Transportation Research Board, pp. 219-224.
4. S. Haykin. Nonlinear Methods of Spectral Analysis. Berlin: Springer-Verlag, 1979.
5. A. V. Oppenheim. Digital Signal Processing. Englewood Cliffs, N. J.: Prentice-Hall, 1975.
6. J. P. Burg. A New Analysis Technique for Time Series Data. Enschede, Holland: Advanced Study Instruction on Signal Processing NARO, 1968.

APPENDIX. THE CONCEPT OF MAXIMUM ENTROPY SPECTRAL ESTIMATION (MESE)

The MESE method was introduced by Burg in 1968 (6). Like maximum likelihood spectral estimation, MESE is a kind of estimator of parameter estimation. Consider a discrete sequence $\{V_i\}$ with sequence length N and sample interval T . If the sequence is a stationary, zero mean, approximately normally distributed, and a band-limited stochastic process, then entropy of the sequence is defined as

$$H = \frac{1}{2} \ln(2B) + \frac{1}{4B} \int_{-B}^B \ln \{ S(\omega) \} d\omega \quad (6)$$

where B is band width of the sequence, and $S(\omega)$ is the spectral density function of the sequence, or

$$S(\omega) = T \sum_{m=-\infty}^{+\infty} R(m) e^{-jmT\omega} \quad (7)$$

In Equation 7, $R(m)$ is defined as the autocorrelation function of sequence $\{V_i\}$

$$R(m) = E \{ V_i V_{i+m} \} \quad (8)$$

Combining Equations 6 and 7, entropy is obtained by

$$H = \frac{1}{2} \ln(2B) + \frac{1}{4B} \int_{-B}^B \ln \left\{ T \sum_{m=-\infty}^{+\infty} R(m) e^{-jmT\omega} \right\} d\omega \quad (9)$$

Suppose the values of autocorrelation $R(m)$ are given for $m = 0, 1, 2, \dots, M$. Then the corresponding extension of the autocorrelation function is defined by the convolution sum

$$R(m) = - \sum_{k=1}^M R(m-k) a_k \quad (m > M) \quad (10)$$

or, equivalently,

$$\sum_{k=0}^M R(m-k) a_k = 0 \quad (a_0 = 1, m > M)$$

The method that Burg introduced maximizes entropy H with respect to $R(m)$ ($|m| > M$) with restrained condition Eq. 10, so that parameters a_1, a_2, \dots, a_m can be obtained. Mathematically, this can be expressed as

$$\left. \begin{aligned} \frac{\partial H}{\partial R(m)} &= 0 & (|m| > M) \\ \sum_{k=0}^M R(m-k) a_k &= 0 \end{aligned} \right\} \quad (11)$$

It can be proved that with the conditions in Eq. 11, sequence $\{V_i\}$ can be related by the following autoregression model [AR(M) model]:

$$V_i = -a_1 V_{i-1} - a_2 V_{i-2} - \dots - a_M V_{i-M} + e_i \quad (12)$$

where M is the order of the AR(M) model, and $\{e_i\}$ is an approximately normally distributed disturbance with zero mean value. Omitting the mathematical derivation, one can obtain the estimate of the parameters (a_1, a_2, \dots, a_m) by the Yule-Welker equation

$$R \cdot A = P \quad (13)$$

where R is the autocorrelation matrix of sequence $\{V_i\}$ and R is called the Toeplitz matrix:

$$R = \begin{bmatrix} R(0) & R(-1) & \dots & R(1-M) & R(-M) \\ R(1) & R(0) & \dots & R(2-M) & R(1-M) \\ \dots & \dots & \dots & \dots & \dots \\ R(M-1) & R(m-2) & \dots & R(0) & R(-1) \\ R(M) & R(M-1) & \dots & R(1) & R(0) \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_{M-1} \\ a_M \end{bmatrix} \quad P = \begin{bmatrix} P_M \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

where

$$P_M = E \{ e_i^2 \}$$

Finally, with all parameters estimated by the MESE algorithm, the maximum entropy spectral density function can be expressed by

$$S(\omega) = \frac{P_M T}{\left| 1 + \sum_{m=1}^M a_m e^{-jmT\omega} \right|^2} \quad (14)$$

