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Abstract

Although splitting shipments across multiple delivery or transportation modes typically increases total shipping costs as a result of diseconomies of scale, it may offer certain benefits that can more than offset these costs. These benefits include a reduction in the probability of stock-out and in the average inventory costs, as well as a concurrent reduction in transportation congestion. We consider a single-stage inventory replenishment model that includes two transportation modes: a cheaper, less reliable mode that is congested, and another, more expensive but perfectly reliable mode. The high-reliability mode is only utilized in replenishment intervals in which the lead time of the less-reliable mode exceeds a certain value. This permits substituting the high-reliability mode for safety stock, to some degree. We characterize optimal replenishment decisions with these two modes, as well as the potential benefits of simultaneously using two delivery modes.

Executive Summary

This project focuses on analysis and validation of the benefits of the strategic use of *mode splitting* in freight delivery. Mode splitting is a form of batch splitting used in inventory control (Thomas and Tyworth, 2006, Minner, 2003). The basic idea of batch splitting is simple and can be described as follows. Suppose a business customer orders Q units of a product from a supplier. When faced with such an order, the supplier often waits until the entire batch is produced, and then ships the Q units to the supplier. Alternatively, the supplier may send smaller shipments before the entire batch is produced (for example, two batches at twice the frequency, each of size $Q/2$). The latter approach implies more frequent deliveries of smaller batch sizes, and may have substantial impacts on a number of factors that affect the total operations costs incurred by both the supplier and customer. Moreover, this splitting strategy may result in negative externalities in the form of increased use of transportation resources and the resulting contribution to traffic congestion.

This research considers how firms make transportation mode choices in freight shipping, how the collective mode selection decisions of these firms contribute to traffic congestion, and whether a more comprehensive view of mode choice decisions can enable both better operations performance for firms and reduced stress on the capacity of transportation networks. Textbook approaches to mode selection decisions employ highly simplified quantitative analysis, considering the average unit costs of transportation and inventory. These approaches typically also make the simplifying assumption that delivery lead times between locations are deterministic, failing to account for uncertainty in transportation times.

This work emphasizes the fact that inventory replenishment and transportation related costs are inter-related, and that inventory replenishment decisions influence transportation costs incurred in making inventory deliveries. Thus, while we focus on the implications of inventory policies, we account for associated transportation costs of different modes and, indirectly, congestion impacts that drive these transportation costs. Our generic use of the term “supply mode” may therefore refer to any of a number of transportation modes.

The goal of this research then is to explore the potential benefits of delivery mode splitting. In order to do this, we will lay a foundation that formalizes the relationships among these variables within a mathematical model. Then, using this model, we will demonstrate how a complete analysis of these factors can lead to decisions that may run counter to current practice, and may provide operational cost performance benefits while reducing the burden on transportation network capacity and congestion.

1 Background

The three main criteria of cost, quality, and delivery reliability are of paramount importance in evaluating supplier performance, where delivery reliability is typically a function of the supplier's lead-time performance, as stated by Minner [10]. All else being equal, lower cost shipping modes imply both longer and less reliable delivery lead-time performance. Using such suppliers therefore tends to necessitate at least an occasional use of quick-response suppliers who can fill the gaps when the low-cost supplier's lead time is too long. Even if a single supplier is preferred for sourcing, a variety of different delivery modes are typically available to shippers, depending on the product type. That is, a wide range of package carriers and third-party logistics providers may be used to fulfill orders. Because of the uncertainty in delivery performance of different modes, it is not immediately clear which mode, or what mix of different modes, should be used when ordering from a supplier or from multiple suppliers.

This research focuses on the benefits of delivery mode diversification as a strategy to reduce inventory replenishment and delivery costs. We consider a system in which a "buyer" uses two different delivery modes for inventory replenishment. These modes may correspond to entirely different suppliers, or they may imply different transportation modes employed by a single supplier. In particular, we consider an inventory replenishment model where the buyer uses a continuous review (Q, r) replenishment policy and has a constant, deterministic demand rate λ. Our model assumes that the buyer does not permit shortages, and may use two different shipping modes within each inventory replenishment cycle. One of these modes comes at a low cost, but is less reliable, reflected by a stochastic delivery lead time (with a finite upper bound). In addition, a higher cost, perfectly reliable shipping mode is available if needed.

We assume that the *uncertainty* in the delivery time of the less reliable supply mode results to a large extent from a high utilization of the associated transportation mode. Our model considers the effect that an increase in the utilization of a stressed shipping or transportation mode has on delivery performance by expressing the less reliable supply mode lead time as a stepwise nondecreasing function of the order size. We can think of the less reliable supply mode as traversing an already congested transportation route, where sending a larger quantity via this mode might require increasing the number of trucks using the route (or the frequency at which trucks utilize the route). As a result, more capacity is consumed and the expected value and variability of the travel time along the route increases. Alternatively, an increase in order quantity would also likely increase the required production time, which leads to an increase the delivery lead time for the order. For the perfectly reliable transportation mode, we assume that it has a sufficient degree of excess capacity, and therefore, it will not be affected by the quantity ordered.

This work emphasizes the fact that inventory replenishment and transportation related costs are inter-related, and that inventory replenishment decisions influence transportation costs incurred in making inventory deliveries. Thus, while we focus on the implications of inventory policies, we account for associated transportation costs of different modes and, indirectly, congestion impacts that drive these transportation costs. Our generic use of the term "supply mode" may therefore refer to any of a number of transportation modes.

Our work falls within a stream of literature on multiple sourcing inventory models. Following the characterization that Minner [10] presented in his review of multiple-supplier inventory models, we identify two main lines of research based on the assumptions regarding supplier lead times. First, we will review inventory models with n supply modes with deterministic lead times, where the general assumption is that $L_1 < L_2 ... < L_n$ and $c_1 > c_2 ... > c_n$, where L_i and c_i are the lead time and unit purchasing cost for supply mode i, respectively. The main focus of most of the works that use this assumptions is to define optimal replenishment policies or to define policy parameters for a specific replenishment policy. After discussing these works, we will review models that assume multiple delivery options with stochastic lead times. Those models are primarily concerned with order splitting, where an order is placed at the beginning of a replenishment cycle and is split among different suppliers.

Among the studies that assume the use of supply modes with deterministic lead times, Moinzadeh and Nahmias [11] consider a model with a continuous review policy, two supply modes and random demand; their goal is to find the order quantities and reorder points for each supply mode in order to minimize total inventory costs. Later, Moinzadeh and Schmidt [12] presented a model based on a one-for-one $(S-1, S)$ inventory policy with two supply options, where the decision on whether to place an order with the regular or emergency supplier depends on the age of the outstanding order. In contrast with our model, these works assume deterministic lead times for both supply modes and allow inventory shortages.

Several past works on multiple sourcing consider a fast, emergency supplier, which can be used to avoid shortages by placing an emergency order (based on some triggering condition). For example, Tagaras and Vlachos [17] assume an emergency order is placed when the probability of stockout is high; Chiang and Gutierrez [3] propose an 'indifference level' for inventory that triggers a second order; in Johansen and Thorstenson [6], the reorder point for the emergency supplier is a function of the time until the regular order arrives. Our model is similar in spirit to these works, except that an order is placed with the 'emergency supplier' only when reaching a given time point in the replenishment cycle, and only if the order from the primary (cheaper) supply mode has not arrived by this time point.

Instead of using multiple supply modes, Chiang and Chiang [2] propose an inventory model with one supplier and multiple deliveries. They assume the use of a continuous review policy with Normally distributed demand, constant lead time and a predetermined service level. In their model an order of size Q is placed at the beginning of the replenishment cycle, which is split between n deliveries with interarrival times L_i for $i = 1, 2, ..n$. We find a similar approach in Chiang [1], who proposes the use of one supply mode and multiple deliveries for periodic review (R, S) inventory systems. In both of these papers, the cost reduction, when compared with a single delivery model, results from a reduction in cycle stock. Unlike the previous works in the literature, we propose the use of multiple supply modes with different ordering and purchasing costs and different levels of delivery reliability. Our model does not place orders with the two supply modes at the beginning of the replenishment cycle; on the contrary, we decide wether to place a second order with the more reliable and expensive supply mode at a specific time during the replenishment cycle, only if the first order has not arrived by that time.

Past works on models that assume stochastic lead times and multiple delivery options are primarily concerned with order splitting. In order splitting, an order is placed at the beginning of the replenishment cycle, and is split among different suppliers. The main benefit of using order splitting is the reduction in the effective lead time (the time until the first order arrives), which leads to a reduction in the safety stock level required to meet a given service level. This effect was demonstrated by Sculli and Wu [16] and Kelle and Silver [7], when each order is split between two suppliers. Later, Pan et al. [13], based on order statistics, presented expressions to estimate the parameters for the distributions of the the effective lead time and time between arrivals when two supply modes with identical lead times distributions are used. As an example, they considered three lead time distributions: Normal, Uniform, and Exponential, and observed a reduction in the the mean and variance of the first effective lead time, compare with the use of a single supply mode.

Most of the relevant models that assume stochastic lead times also assume (Q, r) inventory policies, and seek the optimal order quantity, reorder point and the proportion of order splitting, as in Lau and Lau [8] and Lau and Zhao [9], the latter accounting for stochastic demands in their model. These works showed that most of the savings from order splitting result from a reduction in cycle stock, which often exceeds the cost savings due to associated safety stock reduction. Ramasesh et al. [14] showed that, under the assumption of identical lead time distributions for the different supply modes, an optimal solution is found when the order is split equally, and that the savings come from reduced holding and backordering costs. A different approach was presented by Ganeshan, Tyworth and Guo [5], who proposed a discounting option for the less reliable supplier, and presented exchange curves to define when it is beneficial to split the order, as well as the percentage discount necessary to make the model attractive.

Our model differs from previous literature as it considers one (cheaper, less reliable) supply mode with a stochastic lead time and another with a perfectly reliable (deterministic) lead time. This more closely reflects typical cases in which a buyer uses a primary delivery mode that is not perfectly reliable, but may expedite deliveries (via, e.g., overnight shipping) when stockouts become imminent. In contrast with order splitting models, our work does not place an order with two shipping modes at the beginning of every replenishment cycle. Instead, we seek the optimal reorder point and order quantity such that the long-run expected inventory ordering, holding and purchasing costs are minimized without any shortages (i.e., with a 100% service level). Note that a distinguishing feature of our model is reflected in the fact that we need not always place an order with the reliable delivery mode. That is, if the order from the unreliable supplier arrives sufficiently early in the cycle, we need not place an order with the reliable supply mode. In other words, we can 'wait-and-see' whether a second order is needed in each cycle, allowing us to incorporate specific information about the unreliable supplier's performance in each cycle before determining whether a second order is needed.

In particular, our model sets a reorder point $r = \lambda\bar{\tau}$, where $\bar{\tau}$ is an amount of time that may be less than the upper limit of the lead time the lower cost shipping mode (assuming this lead time distribution is bounded from above). If an order is placed at time zero, then $\bar{\tau}$ is the time at which the current on-hand inventory will reach zero, i.e., the inventory 'run-out' time. Suppose that at the beginning of each replenishment cycle (at time zero for the cycle), the buyer orders from the less reliable shipping mode. If the order has not been received by time $\bar{\tau} - L_2$, where L_2 is the lead time of the perfectly reliable shipping mode, the buyer will need to place a second order with the faster, reliable shipping mode, which will be received at time $\bar{\tau}$, thereby avoiding any shortages.

Another important attribute that differentiates our model from previous works is that we account for the impact that a variation in the utilization level of a congested supply mode has on delivery performance. We assume that the lead time of the less reliable supply mode follows a continuous distribution that is supported on a bounded interval, and that as we increase the size of the order placed with this supplier, the upper bound on its delivery time may also increase; therefore, there is a corresponding increase in the variance of the delivery time.

We are interested in the degree to which using the reliable, quick-response supply mode can reduce inventory-related costs when compared with the use of a single shipping mode. We are also interested in characterizing the circumstances and system parameter values under which the use of both supply modes is beneficial. The reminder of this paper is organized as follows. Section 2 presents our model description, where expressions for the cost functions are derived. Section 3 presents a numerical analysis and shows the circumstances under which our proposed model is preferable to using only a single shipping mode, and Section 4 summarizes our work and discusses future research directions.

2 Research Approach

This section first describes the costs associated with the use of a single shipping mode and then with the use of two shipping modes for inventory replenishment.

2.1 Single-mode Model

We first consider the cost of using each shipping mode independently, which will permit benchmarking the performance of the model with two delivery modes.

2.1.1 Shipping Mode with Uniformly Distributed Lead Time

We consider an inventory replenishment model where a buyer uses a continuous review (Q, r) policy, with constant demand rate λ, where no order crossing is permitted. The buyer orders from a supplier with unlimited capacity who ships via a mode with a stochastic lead time, called shipping mode 1. We assume that this delivery lead time follows a Uniform distribution, where L_1 is the random variable for mode 1 lead time, and τ_l and τ_u denote the lower and upper limits for L_1 (note that our later numerical tests will consider a more general class of lead-time distribution, namely a Beta distribution).

We assume that the buyer desires a zero probability of stockout; therefore, $Q \geq r$ and the reorder point, r, must equal $\lambda\tau_u$, when the supplier uses mode 1 exclusively. Figure 1 shows a realization of the model, assuming without loss of generality that an order is placed at time zero.

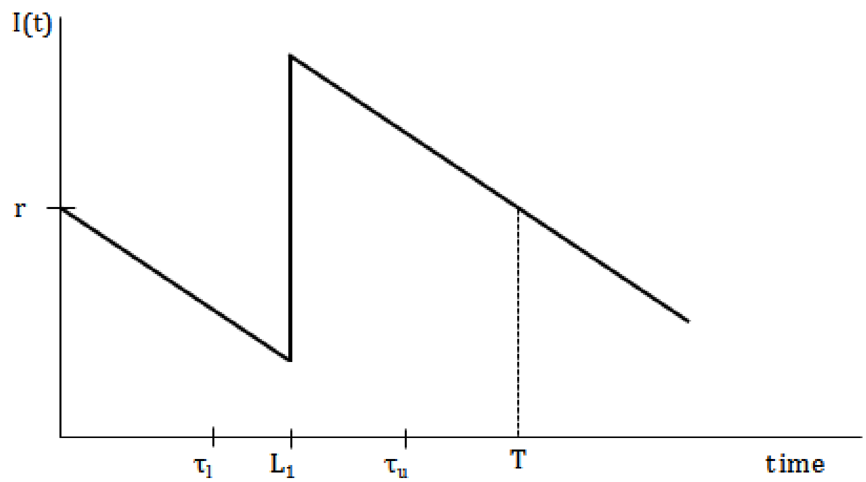

Figure 1: Single-mode model, where $L_1 \sim U(\tau_l, \tau_u)$.

Since the lead time is Uniformly distributed, the probability of an order arrival before or at time t is $P\{L_1 \leq t\} = \frac{t-\tau_l}{\tau_u-\tau_l}$ where $t \in [\tau_l, \tau_u]$. Based on this probability we can compute the expected inventory level at any time $t \in [0, T]$, where $t = 0$ is the time when an order is placed, i.e., when the inventory level is equal to r, and $T = \frac{Q}{\lambda}$ is the length of the replenishment cycle.

To find the expected inventory level at any time $t \in [0, T]$, we divide the cycle into three different intervals: $[0, \tau_l), [\tau_l, \tau_u), [\tau_u, T]$. The first interval starts at time zero, when the inventory level is equal to r and an order of size Q is placed. From this time until the end of the cycle the inventory level will be depleted at a constant rate λ per unit of time due to demand. Since the order will arrive at or after time τ_l, we have that $P\{L_1 < \tau_l\} = 0$, and the expected inventory level during the interval $[0, \tau_l)$ is $\lambda\tau_u - \lambda t$.

Given that the arrival time is Uniformly distributed between τ_l and τ_u, at any time $t \in [\tau_l, \tau_u]$ the expected inventory level increases (with respect to the level during the interval $[0, \tau_l)$) by $QP\{L_1 \le t\}$; therefore, during the interval $[\tau_l, \tau_u)$ the expected inventory level is $\lambda\tau_u + Q\left(\frac{t-\tau_l}{\tau_u-\tau_l}\right) - \lambda t$. For the remaining interval, the order of size Q has been received with probability one, and therefore, the expected inventory level is $\lambda\tau_u + Q - \lambda t$.

As a result, the expected inventory level is:

$$E[I(t)] = \begin{cases} \lambda\tau_u - \lambda t & 0 \le t < \tau_l \\ t\left[\frac{Q}{\tau_u-\tau_l} - \lambda\right] + \left[\lambda\tau_u - \frac{Q\tau_l}{\tau_u-\tau_l}\right] & \tau_l \le t \le \tau_u \\ (Q + \lambda\tau_u) - \lambda t & \tau_u < t \le T \end{cases}$$

As shown in Appendix A we need $Q \ge \lambda(\tau_u - \tau_l)$ to prevent order crossing. Also, in order to prevent shortages, the size of the order has to be at least the size of the reorder point, i.e., $Q \ge \lambda\tau_u$. Based on the previous conditions for the size of Q, we can see that the expected inventory during $t \in [\tau_l, \tau_u]$ will have a positive slope. Figure 2 shows the expected inventory level as a function of time.

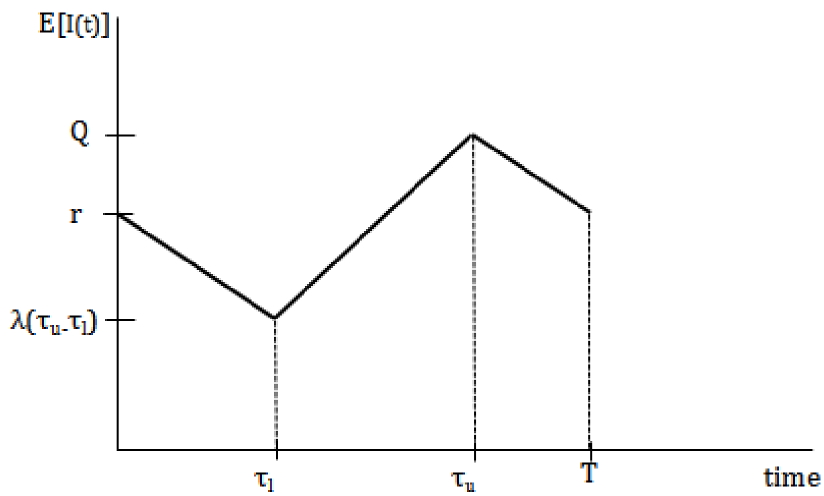

Figure 2: Expected Inventory as a function of time for the single-mode model, where $L_1 \sim U(\tau_l, \tau_u)$.

We next determine the average inventory level in a replenishment cycle. In order to find the average total inventory per cycle, denoted by I_T, we integrate the expected inventory level over the cycle, which is equal to:

$$I_T = \frac{Q^2}{2\lambda} + \frac{Q(\tau_u - \tau_l)}{2}.$$

To obtain the average inventory level per unit time, denoted by $\hat{I}$, we divide the average total inventory by T, which gives

$$\hat{I} = \frac{Q}{2} + \frac{1}{2}\lambda(\tau_u - \tau_l).$$

As we can see, this model has a safety stock of $\frac{1}{2}\lambda\left(\tau_u - \tau_l\right)$ since, on average, orders will be received at time $\frac{1}{2}\left(\tau_u + \tau_l\right)$ and the inventory position at that time is $I\left(t = \frac{\tau_u - \tau_l}{2}\right) = r - \frac{1}{2}\lambda\left(\tau_u + \tau_l\right)$, which is equal to $\frac{1}{2}\lambda\left(\tau_u - \tau_l\right)$.

To compute the total relevant cost per cycle we consider a fixed order cost, inventory holding cost, and variable purchasing cost. We define A_1 as the cost of placing an order using shipping mode 1, h as the inventory holding cost per unit per unit time, and c_1 as the unit purchasing cost when using mode 1. For this model, the average total cost per replenishment cycle is

$$TC_1\left(Q\right) = A_1 + h\left(\frac{Q^2}{2\lambda} + \frac{Q\left(\tau_u - \tau_l\right)}{2}\right) + c_1 Q.$$

The average cost per cycle is equal to the total cost divided by the length of the cycle, which equals

$$G_1\left(Q\right) = A_1\frac{\lambda}{Q} + h\left(\frac{Q}{2} + \frac{1}{2}\lambda\left(\tau_u - \tau_l\right)\right) + c_1\lambda. \tag{2.1}$$

Since the second derivative of the average cost function, $\frac{\partial^2 G_1}{\partial Q^2} = \frac{2A_1\lambda}{Q^3}$, is non-negative $\forall Q > 0$, we conclude that the function is convex for $Q > 0$ and will reach its minimum at Q such that $\frac{\partial f}{\partial Q} = 0$, which implies

$$Q_1^* = \sqrt{\frac{2\lambda A_1}{h}}. \tag{2.2}$$

The above equation implies that the order quantity corresponds to the economic order quantity (EOQ). Using Q_1^* in $G_1\left(Q\right)$ we have a minimum average cost per cycle of

$$G_1 = \sqrt{2\lambda A_1 h} + \frac{\lambda h \Delta\tau}{2} + \lambda c_1, \tag{2.3}$$

where $\Delta\tau = \tau_2 - \tau_1$. We may view the second term above, $\frac{\lambda h \Delta\tau}{2}$, as the incremental cost over that of the standard EOQ Model, due to the uncertainty in the delivery mode 1 lead time.

We next consider the possibility that the degree of uncertainty in the delivery time of shipping mode 1 may be affected by the level of utilization of this mode. Therefore, we would like to account for the potential effect that a change in the utilization of mode 1 has on delivery performance. To approximate this effect, we define the upper limit on mode 1 lead time, τ_u, as a step function of the order size Q, such that, $\tau_u = \tau_u^i$ if $q_i \leq Q < q_{i+1}$ for $i = 1, 2, ..n$ where $q_i < q_{i+1}$, $\tau_u^i < \tau_u^{i+1}$ and n is the number of intervals for Q. Figure 3 shows a graph of τ_u as function of Q.

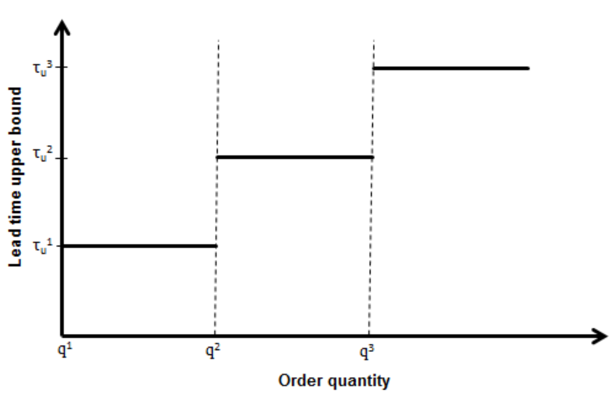

Figure 3: Lead time upper bound as a stepwise function of the order quantity.

The procedure to find the optimal order size Q under the assumption that τ_u is a stepwise function of the order size is similar to the one used by Chopra and Meindl [4] to find the optimal order quantity under an all units quantity discount scheme.

Observe from Equation (2.2) that the optimal order size under supply mode 1 is independent of τ_u. Consequently, as we increase the value of τ_u, all else being equal, the value of Q that minimizes the average inventory cost will not change, but the average inventory cost will increase by $\frac{1}{2}\lambda h\left(\tau_u^{i+1} - \tau_u^i\right)$ for $i = 1, ..n-1$, where n is the number of intervals for τ_u. Figure 4 shows the average inventory cost function when τ_u follows a step function of Q.

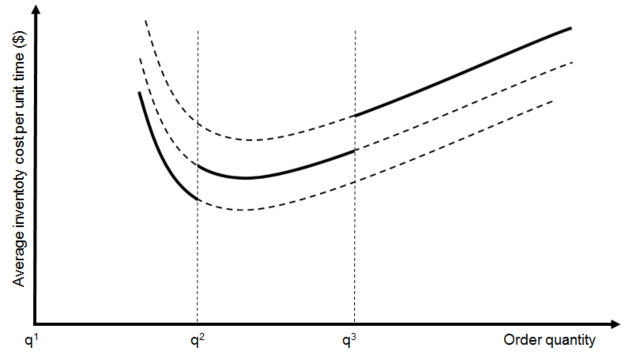

Figure 4: Average inventory cost when lead time upper bound is a function of quantity ordered.

Based on the previous observation, the optimal value of Q when τ_u is a step function of the order size is either on the (unique) interval where Q_1, obtained from (2.2), is feasible, or at one of the break points to the left of Q_1. Appendix B presents a detailed description of the solution procedure to find Q_1^* when shipping mode 1 is used exclusively.

An alternative approach to account for the impact that utilization has on the delivery performance of mode 1 would be to define the lead time upper bound of mode 1 as a linear function of Q, such that $\tau_u = \hat{\tau}_u + \gamma Q$, where $\hat{\tau}_u$ is the base level for the lead time upper bound of mode 1, and γ is the factor by which the lead time increases per unit ordered. The procedure to find the average inventory cost per unit time when τ_u is a linear function of Q, represented by G_1^γ, is similar to the one used when τ_u is fixed. As result we obtain:

$$G_1^\gamma(Q) = A_1 \frac{\lambda}{Q} + h\left(\frac{Q}{2} + \frac{1}{2}\lambda\left(\hat{\tau}_u + \gamma Q - \tau_l\right)\right) + c_1\lambda.$$

Observe that when the lead time upper bound is a linear function of the order size Q, then as we increase the quantity ordered we will have an increase in the average inventory holding cost equal to $\frac{1}{2}h\lambda\gamma Q$, due to the increase in safety stock.

The average inventory cost per unit time is a convex function in Q, since $\frac{\partial^2 G_1^\gamma}{\partial Q^2} = \frac{2A_1\lambda}{Q^3} > 0$ $\forall Q > 0$, the order size that minimizes the average inventory cost is equal to

$$Q^{\gamma *} = \sqrt{\frac{2\lambda A_1}{h\left(1 + \lambda\gamma\right)}},$$

and the minimum average inventory cost is

$$G_1 = \sqrt{2\lambda A_1 h\left(1 + \lambda\gamma\right)} + \frac{h\lambda\Delta\tau}{2} + \lambda c_1.$$

For the remainder, we will assume that the lead time upper bound for delivery mode 1, τ_u, is a step function of Q, since we believe this provides a better approximation of the effect that the order quantity has on the delivery time of a stressed supply mode. That is, a step function allows for changes at discrete quantities, as may be the case when an additional truck is required for a highly congested route or an additional production batch is required on a capacitated production line.

2.1.2 Deterministic Lead Time Mode

We next consider the cost of using a shipping mode, called mode 2, that has a deterministic lead time L_2. In this case we can use the EOQ model where Q_2^* is equal to

$$Q_2^* = \sqrt{\frac{2\lambda A_2}{h}},$$

A_2 is the fixed cost of placing an order when using mode 2, h is the inventory holding cost per unit per unit time, and λ is the constant demand rate. We recognize that a more general model would permit holding costs that depend on the variable purchase cost. However, this difference in holding cost is typically very small in practice, and the resulting model in which holding costs do not depend on purchase cost permits obtaining much more in the way of analytical results.

Letting c_2 denote the unit purchasing cost for mode 2 and $\Delta c = c_2 - c_1$, the average cost per replenishment cycle is equal to:

$$G_2 = \sqrt{2\lambda A_2 h} + \lambda\left(c_1 + \Delta c\right). \tag{2.4}$$

In order to compare the average cost of exclusively using shipping mode 1 versus mode 2, we subtract (2.4) from (2.3). Assuming that $A_1 \leq A_2$ and $\Delta c \geq 0$ we obtain:

$$G_1 - G_2 \leq \frac{\lambda h \Delta\tau}{2} - \lambda \Delta c = \omega, \tag{2.5}$$

where $\omega = \frac{\lambda h \Delta\tau}{2} - \lambda\Delta c$. Note that when $\omega \leq 0$ mode 1 should be chosen as the preferred single mode. Later we will use this expression during the computation of the average inventory cost per replenishment cycle for the dual-mode model.

2.2 Dual-Mode Model

We next present a model where the buyer may use two different shipping modes, implying a positive (expected) order quantity for both modes. We will then compare our proposed dual-mode model with the single-mode model and choose the one with the minimum expected cost per unit time.

Our proposed model assumes that in addition to mode 1, the buyer may place a second order that will use an expedited shipping mode, called mode 2, with deterministic lead time, L_2. In this case, in order to reduce the required safety stock, the buyer may define a different reorder point $r = \lambda\bar{\tau}$, where $\bar{\tau} \in [L_2, \tau_u)$, and where $Q \geq r$ in order to avoid shortages. Note that for the dual-mode model, $\bar{\tau}$ is strictly less than τ_u (in other words, the expected order quantity via mode 2 is strictly positive; the case in which the expected order quantity is zero corresponds to the single-mode case with only mode 1). Using $\bar{\tau}$ to define the reorder point increases the probability of stockout to $P\{L_1 \geq \bar{\tau}\} = \frac{\tau_u - \max\{\tau_l, \bar{\tau}\}}{\tau_u - \tau_l}$, which is strictly greater than zero, since $\bar{\tau} < \tau_u$.

Given that the buyer desires a zero probability of stockout, this may require using mode 2, which will be used as a backup in cases where $L_1 > \bar{\tau} - L_2$. Note that $\bar{\tau} - L_2$ is the time at which the buyer **must** place a second order using mode 2 if the mode 1 order has not arrived. If this is the case, then the buyer must place an order for $\lambda\left(\tau_u - \bar{\tau}\right)$ units that will be sent via mode 2. This second order will arrive at time $\bar{\tau}$, which is the inventory run-out time, and must contribute enough inventory to avoid stocking out in the case that $L_1 = \tau_u$, which is the worst-case delivery time for mode 1.

Note that we assume that τ_u is a step function of Q, as shown in Figure 3. This means that the lead time upper bound for mode 1 will increase for increasing values of the quantity ordered. In the following section we first develop the expressions for the average inventory cost for the dual-mode model, and after that we will present a procedure to find an optimal solution for our proposed model when τ_u is a step function of Q.

Observe that if $\bar{\tau} - L_2 < \tau_l$, then the buyer must decide on whether to place an order via mode 2 before any information regarding the value of L_1 can be obtained (i.e., before time τ_l). In this case, the supplier must place an order with mode 2 (otherwise, the probability of a stockout is positive, violating the 100% service level). On the other hand, if $\bar{\tau} - L_2 \geq \tau_l$, then there is a positive probability that no order will be placed via mode 2 in a cycle. Because there are two distinct cases based on the relative values of $\bar{\tau} - L_2$ and τ_1, the following subsections will consider these cases separately.

2.2.1 CASE 1: $\bar{\tau} \in [L_2, \tau_l + L_2)$

At time $\bar{\tau} - L_2$, the buyer must determine whether the mode 1 order has arrived to evaluate whether it is necessary to place a mode 2 order. When $\bar{\tau} \in [L_2, \tau_l + L_2)$ we have that $\bar{\tau} - L_2 < \tau_l$, and, therefore, the buyer will always place a mode 2 order. Figure 5 shows an example of the model realization.

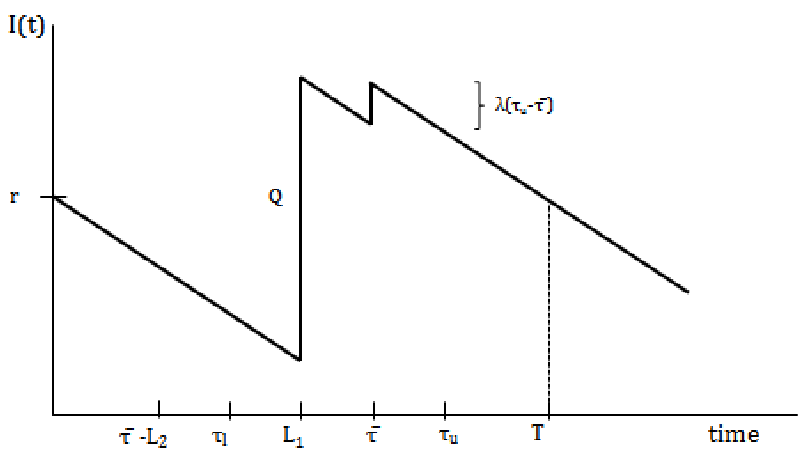

Figure 5: Dual-mode model realization. Case 1: $\bar{\tau} \in [L_2, \tau_l + L_2)$.

Observe that the replenishment cycle starts when the inventory level reaches $r = \lambda\bar{\tau}$ units. During the cycle the buyer orders Q units via mode 1 and $\lambda\left(\tau_u - \bar{\tau}\right)$ units via mode 2. Therefore, the cycle will restart at time $T = \frac{Q}{\lambda} + (\tau_u - \bar{\tau})$.

To analyze the expected inventory level at any time t, we divide the replenishment cycle into four intervals: $[0, \tau_l), [\tau_l, \bar{\tau}), [\bar{\tau}, \tau_u), [\tau_u, T]$. Assume without loss of generality that the cycle starts at time $t = 0$, when the inventory level is $\lambda\bar{\tau}$ and an order of size Q is placed with mode 1. This order will be received at some time $t \in [\tau_l, \tau_u]$; therefore, the probability of receiving the order during the first interval is $P\{L_1 < \tau_l\} = 0$.

During the replenishment cycle, the inventory level will decrease at a constant rate λ per unit time due to demand, and hence the expected inventory level during $t \in [0, \tau_l)$ is $\lambda\bar{\tau} - \lambda t$.

During the second interval $t \in [\tau_l, \bar{\tau})$, the expected inventory level is equal to $\lambda\bar{\tau} + QP\{L_1 \le t\} - \lambda t$, where $P\{L_1 \le t\} = \frac{t-\tau_l}{\tau_u-\tau_l}$, as in the single-mode model.

A shift in the expected inventory function occurs at $t = \bar{\tau}$, when the order of size $\lambda(\tau_u - \bar{\tau})$ shipped via mode 2 arrives, and hence the expected inventory level during the third interval is $\lambda\tau_u + QP\{L_1 \le t\} - \lambda t$.

By time $t = \tau_u$ the order from supplier 1 has arrived with probability one and the expected inventory level is $Q + \lambda\tau_u - \lambda t$ until the end of the cycle.

A summary of the expected inventory level as a function of time is presented below.

$$E[I(t)] = \begin{cases} \lambda\bar{\tau} - \lambda t & 0 \le t < \tau_l \\ t\left[\frac{Q}{\tau_u-\tau_l} - \lambda\right] + \left[\lambda\bar{\tau} - \frac{Q\tau_l}{\tau_u-\tau_l}\right] & \tau_l \le t < \bar{\tau} \\ t\left[\frac{Q}{\tau_u-\tau_l} - \lambda\right] + \left[\lambda\tau_u - \frac{Q\tau_l}{\tau_u-\tau_l}\right] & \bar{\tau} \le t < \tau_u \\ (Q + \lambda\tau_u) - \lambda t & \tau_u \le t \le \frac{Q}{\lambda} + (\tau_u - \bar{\tau}) \end{cases}$$

Figure 6 shows the expected inventory level as a function of time.

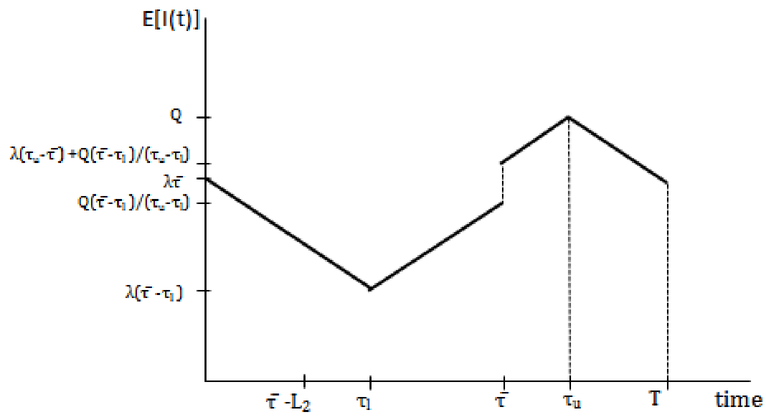

Figure 6: Expected Inventory as function of time, Case 1: $\bar{\tau} \in [L_2, \tau_l + L_2)$.

Recall that because we assume $Q \ge \lambda(\tau_u - \tau_l)$ (that is, we assume no order crossing; see Appendix A), the slope of the function for $t \in [\tau_l, \tau_u]$ is non-negative.

We can integrate the expected inventory level curve over the cycle to find the average total inventory per cycle, which gives

$$I_T = \frac{Q^2}{2\lambda} + \frac{Q(\tau_u - \tau_l)}{2} + \frac{\lambda}{2}(\tau_u - \bar{\tau})^2 .$$

The average inventory per cycle is the average total inventory divided by T, which equals

$$\hat{I} = \frac{Q^2 + Q\lambda(\tau_u - \tau_l) + \lambda^2(\tau_u - \bar{\tau})^2}{2[Q + \lambda(\tau_u - \bar{\tau})]}.$$

For the dual-mode model, the fixed order cost has two components: A_1, which is the fixed order cost incurred by the buyer when placing the first order via mode 1; and $\bar{A}$, which is the additional order cost incurred when placing an expedited order via mode 2. Note that the expedited order may be placed with the same supplier or may be planned when placing the first order; we therefore assume that $\bar{A} \leq A_2$, since part of the associated fixed order cost is incurred when placing the first order in the cycle.

The total cost per cycle includes the fixed order cost, $A_1 + \bar{A}$, the inventory holding cost per unit per unit time, h, and the purchasing cost per unit shipped by modes 1 and 2, denoted by c_1 and c_2, respectively. This implies an average total cost per cycle equal to:

$$TC_I = \left(A_1 + \bar{A}\right) + h\left(\frac{Q^2}{2\lambda} + \frac{Q\left(\tau_u - \tau_l\right)}{2} + \frac{\lambda}{2}\left(\tau_u - \bar{\tau}\right)^2\right) + \left(Qc_1 + \lambda\left(\tau_u - \bar{\tau}\right)c_2\right).$$

The average cost per cycle is equal to the total cost per cycle divided by the cycle length T, which equals

$$G_I\left(Q, \bar{\tau}\right) = \frac{\lambda\left(A_1 + \bar{A}\right) + \frac{h}{2}\left(Q^2 + Q\lambda\left(\tau_u - \tau_l\right) + \lambda^2\left(\tau_u - \bar{\tau}\right)^2\right) + c_1 Q\lambda + c_2\lambda^2\left(\tau_u - \bar{\tau}\right)}{Q + \lambda\left(\tau_u - \bar{\tau}\right)}.$$

Without loss of generality we let $A = A_1 + \bar{A}$ and $\beta = (\tau_2 - \bar{\tau})$. Observe that $\lambda\beta = Q_2$, i.e., the quantity delivered via mode 2. Thus, the total order quantity in a cycle equals $Q + \lambda\beta$.

Noting that Q and β are decision variables, the average cost per unit time is given by

$$G_I\left(Q, \beta\right) = \frac{\lambda A}{\left(Q + \lambda\beta\right)} + \frac{h\left(Q + \lambda\beta\right)}{2} + \frac{\lambda h Q}{\left(Q + \lambda\beta\right)}\left(\frac{\Delta\tau}{2} - \beta\right) + \frac{\lambda}{\left(Q + \lambda\beta\right)}\left(c_1 Q_1 + c_2\lambda\beta\right). \quad (2.6)$$

From Appendix C, we have that $G_I\left(Q, \beta\right)$ is convex $\forall Q > 0$ and $\beta > 0$ if $4\lambda A h > \omega^2$, where $\omega = \frac{\lambda h \Delta\tau}{2} - \lambda\Delta c$ (Appendix C also provides an argument as to why this condition is likely to be mild and non-restrictive in practice). Therefore we find a stationary point for $G_I\left(Q, \beta\right)$ using the conditions $\frac{\partial G_I}{\partial \beta} = 0$ and $\frac{\partial G_I}{\partial Q} = 0$.

As result, we have that for a given β, assuming the convexity condition is met, the optimal Q and minimum average inventory cost per cycle are:

$$Q^*\left(\beta\right) = -\lambda\beta + \sqrt{2\lambda^2\beta^2 - \lambda^2\beta\Delta\tau + \frac{2\lambda^2\beta\Delta c}{h} + \frac{2\lambda A}{h}} \quad (2.7)$$

$$G_I\left(\beta\right) = h\sqrt{2\lambda^2\beta^2 - \lambda^2\beta\Delta\tau + \frac{2\lambda^2\beta\Delta c}{h} + \frac{2\lambda A}{h}} + \lambda h\left(\frac{\Delta\tau}{2} - \beta\right) + \lambda c_1 \quad (2.8)$$

Using $\frac{\partial G_I}{\partial \beta} = 0$, we arrive at the following stationary point solution for β:

$$\beta^* = \frac{1}{2\lambda h}\left[\sqrt{4\lambda A h - \left(\frac{\lambda h \Delta\tau}{2} - \lambda\Delta c\right)^2} + \left(\frac{\lambda h\Delta\tau}{2} - \lambda\Delta c\right)\right].$$

Substituting $\omega = \frac{\lambda h\Delta\tau}{2} - \lambda\Delta c$ and $\alpha = 4\lambda A h$, we have

$$\beta^* = \frac{1}{2\lambda h}\left[\sqrt{\alpha - \omega^2} + \omega\right], \quad (2.9)$$

and the optimal value for Q can then be expressed as

$$Q^* = \frac{1}{2h}\left[\sqrt{\alpha - \omega^2} - \omega\right]. \quad (2.10)$$

Assuming the stationary point solution is feasible, because the total order quantity per cycle equals $Q^* + \lambda\beta^*$, Equations (2.9) and (2.10) indicate that the total order quantity per cycle at optimality equals

$$Q^* + \lambda\beta^* = \frac{\sqrt{\alpha - \omega^2}}{h}. \tag{2.11}$$

Based on (2.9) through (2.11), we observe that the quantities shipped via modes 1 and 2 (and the total order quantity per cycle) are concave functions of ω. The total order quantity per cycle will attain its maximum when $\omega = 0$, which results in $Q^* = \lambda\beta$. This means that when the incremental holding cost due to the uncertainty in the lead time of mode 1 is equal to the additional purchasing cost per unit of mode 2, the dual-mode model will order the same quantity from both supply modes, assuming the stationary points for (2.6) are feasible, i.e., $Q > 0$ and $\beta \in [\tau_u - \tau_l - L_2, \tau_u - L_2)$.

We also observe that when $\omega < 0$ we will have $Q^* > \lambda\beta$. The reason for this is that a negative value of ω indicates that the average inventory cost when using shipping mode 1 exclusively is less than the average inventory cost of using shipping mode 2, and therefore, the dual model model will increase the use of shipping mode 1. The opposite happens when $\omega > 0$, when the incremental cost of using shipping mode 2 over the standard EOQ model is smaller than the incremental cost of using shipping mode 1, and as a result $\lambda\beta > Q^*$.

Using (2.9) and (2.10), the minimum average inventory cost is

$$G_I^* = \frac{1}{2}\left[\sqrt{\alpha - \omega^2} + \omega\right] + \lambda c_2. \tag{2.12}$$

Note that when the stationary point solution β^* is feasible, i.e., $\beta^* \in (\tau_u - \tau_l - L_2, \tau_u - L_2]$ we will use (2.10) and (2.12) for Q^* and G_I^*, respectively; otherwise the optimal value of β will be at an end point of the interval and we will use (2.7) and (2.8) at the appropriate value of β to determine Q^* and G_I^*, respectively.

Now that we have an expression for the average inventory cost for the dual-mode model when $\bar{\tau} \in [L_2, L_2 + \tau_1)$, we can derive a method for determining optimal order quantities for modes 1 and 2 when τ_u is a step function of Q, such that $\tau_u = \tau_u^i$ if $q_i \le Q < q_{i+1}$ for $i = 1, 2, ..n$, where $q_i < q_{i+1}$, $\tau_u^i < \tau_u^{i+1}$, and n is the number of intervals for Q.

Note that because an increase in τ_u increases the uncertainty in the arrival time of the order placed with mode 1, we expect to observe an increase in the average inventory holding cost for increasing values of τ_u. However, in some cases, when the stationary point solution for β is not feasible, and therefore, the optimal β is found at one of the end points of the interval (in particular, $\beta = \tau_u - L_2$), the average inventory cost may actually decrease as τ_u increases. The reason for this is that, in this case, it is preferable to increase the size of the order placed with mode 2, $(\lambda(\tau_u - \bar{\tau}))$; as a result, as we increase the value of τ_u, an increased order quantity with supply mode 2 may actually decrease the average inventory holding cost.

As in the single-mode case, our solution procedure is similar to the one used by Chopra and Meindl in [4] to find the optimal order quantity under the all units quantity discount scheme; however, because G_I may increase or decrease for increasing values of τ_u, as explained above, when we analyze an interval i, if the stationary point for Q is not feasible for that interval, we still need to evaluate the value of Q at one of the breakpoints, depending on whether $Q > q_{i+1}$ or $Q < q_i$. Algorithm 1, in Appendix D, provides a detailed description of the solution procedure to find the minimum average inventory cost for the dual-mode model when τ_u is a step function of Q.

2.2.2 CASE 2: $\bar{\tau} \in [\tau_l + L_2, \tau_u)$

In our dual-mode model, at time $\bar{\tau} - L_2$ the buyer must determine whether or not to place a second order via mode 2 if the first order has not yet arrived. Since Case 2 considers the interval

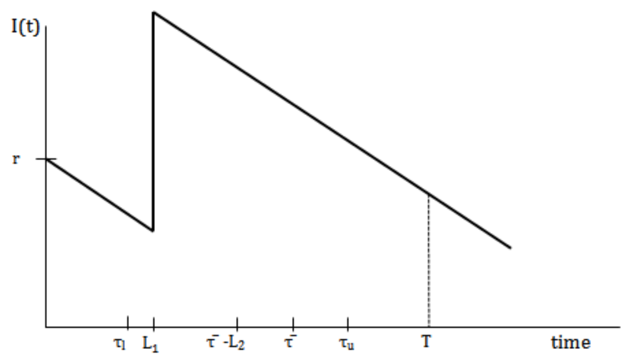

Figure 7: Dual-mode model realization (Case 2.1: $\bar{\tau} \in [\tau_l + L_2, \tau_u)$ and $L_1 \leq \bar{\tau} - L_2$).

$\bar{\tau} \in [\tau_l + L_2, \tau_u)$, we have that $\bar{\tau} - L_2 \geq \tau_l$, and the probability of placing a second order is $P\{L_1 \geq (\bar{\tau} - L_2)\} = \frac{\tau_u - (\bar{\tau} - L_2)}{\tau_u - \tau_l}$.

When $L_1 \leq \bar{\tau} - L_2$ there is no need to place an expedited order using mode 2, and therefore the total amount ordered in the cycle is Q. This occurs with probability $P\{L_1 \leq (\bar{\tau} - L_2)\} = \frac{(\bar{\tau} - L_2 - \tau_1)}{\tau_u - \tau_l}$. Figure 7 shows an example of this case.

When $L_1 > \bar{\tau} - L_2$, a second order of size $\lambda(\tau_u - \bar{\tau})$ is placed at time $t = \bar{\tau} - L_2$, which will be shipped using mode 2 and will be received at time $\bar{\tau}$. This occurs with probability $P\{L_1 > (\bar{\tau} - L_2)\} = \frac{\tau_u - (\bar{\tau} - L_2)}{\tau_u - \tau_l}$. Figure 8 shows an example of this case.

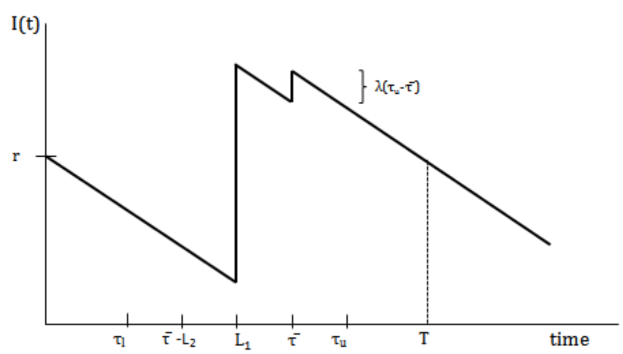

Figure 8: Dual-mode model realization (Case 2.2: $\bar{\tau} \in [\tau_l + L_2, \tau_u)$ and $L_1 > \bar{\tau} - L_2$).

Based on the probability of each of these sub-cases, we can compute the expected inventory as a function of time. As in the previous section we will divide the replenishment cycle into four intervals: $[0, \tau_l), [\tau_l, \bar{\tau}), [\bar{\tau}, \tau_u), [\tau_u, T]$, where T, the cycle length, is now a random variable, and $E[T]$ is the expected cycle length, i.e.,

$$E[T] = \frac{Q}{\lambda} P\{L_1 \leq \bar{\tau} - L_2\} + \left(\frac{Q}{\lambda} + (\tau_u - \bar{\tau})\right) P\{L_1 > \bar{\tau} - L_2\},$$

where, after substituting $P\{L_1 \leq \bar{\tau} - L_2\} = \left(\frac{\bar{\tau} - L_2 - \tau_l}{\tau_u - \tau_l}\right)$, we have

$$E[T] = \frac{Q}{\lambda} + (\tau_u - \bar{\tau}) \left(\frac{\tau_u - \bar{\tau} + L_2}{\tau_u - \tau_l}\right). \tag{2.13}$$

The replenishment cycle starts at time $t = 0$, when the inventory level is $\lambda\bar{\tau}$ and an order of size Q is placed, which will be sent via mode 1 and received at some time $t \in [\tau_l, \tau_u]$; the probability of receiving the order during the first interval is $P\{L_1 < \tau_l\} = 0$.

During the replenishment cycle, the inventory level decreases at a constant rate of λ per unit time due to demand, and hence the expected inventory level during $t \in [0, \tau_l)$ is $\lambda\bar{\tau} - \lambda t$. During the second interval, $t \in [\tau_l, \bar{\tau})$, the expected inventory level is equal to $\lambda\bar{\tau} + QP\{L_1 \leq t\} - \lambda t$, where $P\{L_1 \leq t\} = \frac{t-\tau_l}{\tau_u-\tau_l}$. If, at time $t = \bar{\tau} - L_2$, the buyer does not place an expedited order using mode 2, this means that the mode 1 order has been received, and the inventory level for $t \in [\bar{\tau}, \tau_u)$ is equal to $\lambda\bar{\tau} + Q - \lambda t$, which occurs with probability $P\{L_1 \leq \bar{\tau} - L_2\}$. With probability $P\{L_1 > \bar{\tau} - L_2\}$, however, the buyer places a second order via mode 2 at time $t = \bar{\tau} - L_2$ for $\lambda(\tau_u - \bar{\tau})$ units, which will be received at time $\bar{\tau}$.

For the latter case, when the second order with mode 2 was placed at time $t = \bar{\tau} - L_2$, we can have two situations for the interval $t \in [\bar{\tau}, \tau_u)$:

1. $L_1 > t$, where the inventory level is $\lambda\bar{\tau} + \lambda(\tau_u - \bar{\tau}) - \lambda t$, for $t \in [\bar{\tau}, \tau_u)$;
2. $L_1 \leq t$, where the inventory level is $\lambda\bar{\tau} + \lambda(\tau_u - \bar{\tau}) + Q - \lambda t$, for $t \in [\bar{\tau}, \tau_u)$.

Therefore, for $t \in [\bar{\tau}, \tau_u)$, the expected inventory level is:

$$\begin{aligned} E\left[I\left(t|\bar{\tau} \leq t < \tau_u\right)\right] = &P\{L_1 \leq \bar{\tau} - L_2\}(\lambda\bar{\tau} + Q - \lambda t) + P\{L_1 > t\}(\lambda\bar{\tau} + \lambda(\tau_u - \bar{\tau}) - \lambda t) \\ &+ P\{\bar{\tau} - L_2 < L_1 \leq t\}(\lambda\bar{\tau} + \lambda(\tau_u - \bar{\tau}) + Q - \lambda t), \end{aligned}$$

which results in

$$E\left[I\left(t|\bar{\tau} \leq t < \tau_u\right)\right] = t\left(\frac{Q}{\tau_u - \tau_l} - \lambda\right) + \left(\lambda\bar{\tau} - \frac{Q\tau_l}{\tau_u - \tau_l} + \lambda(\tau_u - \bar{\tau})\left(\frac{\tau_u - \bar{\tau} + L_2}{\tau_u - \tau_l}\right)\right).$$

The last interval of the cycle is $t \in [\tau_u, T]$. By time $t = \tau_u$ the order from supplier 1 has arrived with probability one, and we have two cases with respect to the second order:

1. If $L_1 \leq \bar{\tau} - L_2$, the inventory level is $Q + \lambda\bar{\tau} - \lambda t$, after time $t = \tau_u$, and $T = Q/\lambda + \bar{\tau}$;
2. If $L_1 > \bar{\tau} - L_2$, the inventory level is $Q + \lambda\bar{\tau} + \lambda(\tau_u - \bar{\tau}) - \lambda t$, after time $t = \tau_u$, and $T = Q/\lambda + \tau_u$.

Therefore the expected inventory for $t \in [\tau_u, T]$ is:

$$E[I(t)] = Q + \lambda\bar{\tau} + \lambda(\tau_u - \bar{\tau})\left(\frac{\tau_u - \bar{\tau} + L_2}{\tau_u - \tau_l}\right) - \lambda t$$

A summary of the expected inventory level as a function of time is presented below.

$$E[I(t)] = \begin{cases} \lambda\bar{\tau} - \lambda t & 0 \leq t < \tau_l \\ t\left[\frac{Q}{\tau_u-\tau_l} - \lambda\right] + \left[\lambda\bar{\tau} - \frac{Q\tau_l}{\tau_u-\tau_l}\right] & \tau_l \leq t < \bar{\tau} \\ t\left(\frac{Q}{\tau_u-\tau_l} - \lambda\right) + \left(\lambda\bar{\tau} - \frac{Q\tau_l}{\tau_u-\tau_l} + \lambda(\tau_u - \bar{\tau})\left(\frac{\tau_u-\bar{\tau}+L_2}{\tau_u-\tau_l}\right)\right) & \bar{\tau} \leq t < \tau_u \\ Q + \lambda\bar{\tau} + \lambda(\tau_u - \bar{\tau})\left(\frac{\tau_u-\bar{\tau}+L_2}{\tau_u-\tau_l}\right) - \lambda t & \tau_u \leq t \leq T \end{cases}$$

Figure 9 shows the expected inventory level as a function of time.

The assumptions that the model does not allow shortages and order crossing require that $Q \geq \lambda\bar{\tau}$ and $Q \geq \lambda(\tau_u - \tau_l)$ (from Appendix A), respectively. Therefore, we can see that the slope of the expected inventory function is non-negative for $t \in [\tau_l, \tau_u]$.

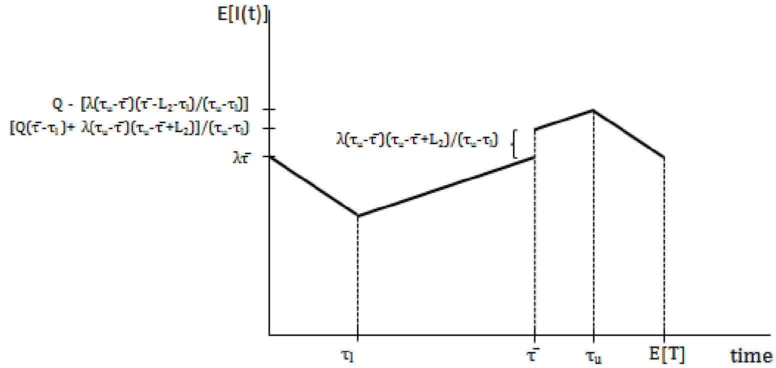

Figure 9: Expected inventory as function of time (Case 2: $\bar{\tau} \in [\tau_l + L_2, \tau_u)$).

In order to characterize the average total inventory per replenishment cycle, we integrate the expected inventory function over the four intervals, and we obtain:

$$E[I] = \frac{Q^2}{2\lambda} + \frac{Q}{2(\tau_u - \tau_l)} \left[(\tau_u - \bar{\tau})^2 + 2(\tau_u - \bar{\tau}) L_2 + (\bar{\tau} - \tau_l)^2 \right] + \frac{\lambda \left[(\tau_u - \bar{\tau})^2 + (\tau_u - \bar{\tau}) L_2 \right]^2}{2(\tau_u - \tau_l)^2} \tag{2.14}$$

Our model can be analyzed as a renewal reward process (see Appendix F), and therefore, the average inventory per unit time, $\frac{E[I(t)]}{t}$ as $t \to \infty$, is equal to $\frac{E[I]}{E[T]}$. Using (2.13) and (2.14) we have that the average inventory per unit time is:

$$\hat{I} = \frac{Q^2 (\tau_u - \tau_l)^2 + \lambda (\tau_u - \tau_l) Q \left[(\tau_u - \bar{\tau})^2 + 2(\tau_u - \bar{\tau}) L_2 + (\bar{\tau} - \tau_l)^2 \right] + \lambda^2 \left[(\tau_u - \bar{\tau})^2 + (\tau_u - \bar{\tau}) L_2 \right]^2}{2 \left[Q (\tau_u - \tau_l)^2 + \lambda (\tau_u - \tau_l) \left[(\tau_u - \bar{\tau})^2 + (\tau_u - \bar{\tau}) L_2 \right] \right]}$$

The expected total cost per cycle is composed of the expected fixed order cost, expected holding cost and expected purchasing cost.

Using A_1 as the cost of placing an order via mode 1 and $\bar{A}$ as the incremental cost of placing a second order via mode 2, the expected fixed order cost is:

$$A_1 P\{L_1 \leq \bar{\tau} - L_2\} + \left(\bar{A} + A_1\right) P\{L_1 > \bar{\tau} - L_2\}.$$

Substituting the probability values, we obtain an expected fixed order cost of

$$A_1 + \bar{A} \left(\frac{\tau_u - \bar{\tau} + L_2}{\tau_u - \tau_1} \right).$$

The expected total inventory holding cost per cycle is

$$h \left\{ \frac{Q^2}{2\lambda} + \frac{Q}{2(\tau_u - \tau_l)} \left[(\tau_u - \bar{\tau})^2 + 2(\tau_u - \bar{\tau}) L_2 + (\bar{\tau} - \tau_l)^2 \right] + \frac{\lambda \left[(\tau_u - \bar{\tau})^2 + (\tau_u - \bar{\tau}) L_2 \right]^2}{2(\tau_u - \tau_l)^2} \right\}.$$

The expected purchasing cost in a cycle is given by

$$c_1 Q P\{L_1 \leq \bar{\tau} - L_2\} + (c_1 Q + c_2 \lambda (\tau_u - \bar{\tau})) P\{L_1 > \bar{\tau} - L_2\}.$$

Substituting the probability values, we obtain an expected variable purchase cost per cycle of

$$Qc_1 + \lambda(\tau_u - \bar{\tau})\, c_2 \left(\frac{\tau_u - \bar{\tau} + L_2}{\tau_u - \tau_1}\right).$$

Adding the expected fixed order cost, expected holding cost and expected purchase cost, we obtain the following expression for the total cost per cycle:

$$\begin{aligned} TC_{II} = {} & A_1 + \bar{A}\left(\frac{\tau_u - \bar{\tau} + L_2}{\tau_u - \tau_l}\right) + Qc_1 + \lambda(\tau_u - \bar{\tau})\, c_2 \left(\frac{\tau_u - \bar{\tau} + L_2}{\tau_u - \tau_l}\right) \\ & + h\left[\frac{Q^2}{2\lambda} + \frac{Q}{2(\tau_u - \tau_l)}\left[(\tau_2 - \bar{\tau})^2 + 2(\tau_u - \bar{\tau})L_2 + (\bar{\tau} - \tau_l)^2\right] + \frac{\lambda\left[(\tau_u - \bar{\tau})^2 + (\tau_u - \bar{\tau})L_2\right]^2}{2(\tau_u - \tau_l)^2}\right]. \end{aligned}$$

The expected cost per unit time, which we denote by G_{II}, is equal to the expected total cost per cycle divided by the expected cycle length, and we therefore have:

$$\begin{aligned} G_{II}(Q,\beta) = {} & \frac{2\lambda A_1 \Delta\tau^2 + 2\lambda A_2 \Delta\tau(\beta + L_B) + 2\lambda\Delta\tau^2 Q c_1}{2\left[Q\Delta\tau^2 + \lambda\Delta\tau(\beta^2 + \beta L_B)\right]} \\ & + \frac{2\lambda^2\Delta\tau\beta(\beta + L_B)c_2 + hQ^2\Delta\tau^2 + \lambda^2 h\left(\beta^2 + \beta L_B\right)^2}{2\left[Q\Delta\tau^2 + \lambda\Delta\tau(\beta^2 + \beta L_B)\right]} \\ & + \frac{h\lambda\Delta\tau Q\left[\beta^2 + 2\beta L_B + (\Delta\tau - \beta)^2\right]}{2\left[Q\Delta\tau^2 + \lambda\Delta\tau(\beta^2 + \beta L_B)\right]}, \end{aligned} \tag{2.15}$$

where $\beta = \tau_u - \bar{\tau}$, $\Delta\tau = \tau_u - \tau_l$, and $\Delta c = c_2 - c_1$. Note that $\lambda\beta$ is the quantity ordered via mode 2 when this order is placed. The decision variables in the above expected cost equation are therefore Q and β.

In order to show the convexity of G_{II} as function of Q, we consider its second derivative, which is given by

$$\begin{aligned} \frac{\partial^2 G_{II}}{\partial Q^2} = {} & \frac{\lambda^2 h\Delta\tau\left(\beta^2 + \beta L_2\right)\left(2\beta\Delta\tau - \Delta\tau^2\right) + 2\lambda^2\Delta\tau^2\left(\beta^2 + \beta L_2\right)\Delta c}{\left[Q\Delta\tau + \lambda(\beta^2 + \beta L_2)\right]^3} \\ & + \frac{2\lambda\Delta\tau^2(\Delta\tau A_1 + A_2(\beta + L_2))}{\left[Q\Delta\tau + \lambda(\beta^2 + \beta L_2)\right]^3}. \end{aligned}$$

It is straightforward to show that the denominator of $\frac{\partial^2 G_{II}}{\partial Q^2}$ is positive $\forall Q > 0$ and $\beta > 0$, and the numerator is non-negative if:

$$\frac{A_1 + \bar{A}p}{\beta p} \geq \omega - \lambda h\beta, \tag{2.16}$$

where $p = \frac{\tau_u - \bar{\tau} + L_2}{\Delta\tau}$ is the probability that an order is placed via mode 2. Note that (2.16) presents a convexity condition for G_{II} as a function of Q for any β, and therefore we need to find the value Q^*_β for minimizing $G^\beta_{II}(Q)$ for any given β, where $\beta \in (0, \Delta\tau - L_2]$ (using, e.g., gradient search techniques). Then, in order to find the optimal values of Q^* and β^* we define $G^*_{II} = \min_{Q>0}\left\{G_{II}\left(Q^*_\beta, \beta\right) \quad \forall\beta \in (0, \Delta\tau - L_2]\right\}$. Although β is a continuous variable that corresponds to the length of time that the order shipped through mode 2 should cover, we will assume that it is sufficient to consider a discrete set of candidate values for β (typically in practice the value of β will be measured in some discrete measure of time, such as days).

We can see that the convexity condition for $G_{II}(Q)$ is not restrictive by observing that whenever $\omega \leq 0$, i.e., when mode 1 is the preferred single mode, then (2.16) will always hold. We also note that the left-hand side of (2.16) is the expected order cost in a cycle divided by the expected extra time in the cycle if an order is placed via mode 2; the left-hand side is therefore an upper bound on the fixed order cost per unit time. The right-hand side of the inequality is less than ω, the amount by which the cost of buffering the uncertainty using mode 1 exceeds the cost of buffering the uncertainty using mode 2. Therefore, even if mode 1 is not the preferred single mode of operation, it is not unlikely for condition (2.16) to hold for a broad range of practical cases.

Since we assume that τ_u is a step function of Q such that, $\tau_u = \tau_u^i$ if $q_i \leq Q < q_{i+1}$ for $i = 1, 2, ..n$ where $q_i < q_{i+1}$, $\tau_u^i < \tau_u^{i+1}$ and n is the number of intervals for Q, we follow the same procedure to find the minimum average cost that we developed in the previous section. In Algorithm 2 (see Appendix E) we present the procedure to find the minimum average inventory cost for the dual-mode model when $\bar{\tau} \in [\tau_1 + L_2, \tau_1 + L_2]$.

3 Findings and Applications

This section reports results of a set of numerical tests that characterize the benefits of the dual-mode model. While we show numerous cases in which the dual-mode model outperforms the better of the two single mode solutions, we also analyze how changes in the model parameters affect the average inventory cost of the dual-mode model (recall that the minimum dual-mode average inventory cost equals $G^* = \min\{G_I^*, G_{II}^*\}$).

Since the average purchasing cost per year is at least λc_1 (whether we use the single or dual-mode model), to compare the results of the dual-mode model against the single-mode model, we subtract λc_1 from the average inventory cost for both models. Therefore we will compare the percentage of cost reduction over the inventory costs that can be modified when using an alternative ordering policy. Hence, the percentage cost reduction is given by:

$$\%CR = \frac{\min\{G_1, G_2\} - G^*}{\min\{G_1, G_2\} - \lambda c_1} \times 100\%,$$

where G_i is the minimum average inventory cost per unit time when the order is shipped by mode i exclusively, $i = \{1, 2\}$ and G^* is the minimum average inventory cost for the dual-mode model. We are of course particularly interested in characterizing conditions under which $\%CR > 0$.

For our numerical analysis, we created problem instances with parameter values similar to the cost structures that can be observed in practice, and also, where we can demonstrate the different effects that changes in a particular parameter may have for the dual-mode model. Table 1 shows the base parameter values used for the numerical analysis of our proposed model.

λ $units/yr$	τ_l $days$	L_2 $days$	h $\$/units \cdot yr$	c_1 $\$/unit$	A_1 $\$/order$	$\bar{A}$ $\$/order$	A_2 $\$/order$
10,000	14	5	1.5	10	100	70	170
τ_u^1 $days$	τ_u^2 $days$	τ_u^3 $days$	τ_u^4 $days$	q^1 $units$	q^2 $units$	q^3 $units$	q^4 $units$
50	55	60	65	0	1,000	2,000	3,000

Table 1: Base parameter values for numerical analysis.

Note that for the fixed order cost, we assume that the fixed cost for mode 1, A_1, is less than or equal to the fixed order cost of mode 2, A_2. Also, assuming that the orders can be placed with the same supplier, but using different shipping modes, we have that $A_1 + \bar{A} \leq A_1 + A_2$, where $\bar{A}$ is the additional order cost of placing a second expedited order, since a portion of the fixed order cost was already incurred when the first order was placed.

Before analyzing the effects of the key parameters on the dual-mode model, we discuss some properties of the optimal solution for our proposed model. As we will see, based on the results in Tables 2 through 7, when $\Delta c = 0$ and $G_1 < G_2$, we have that $\bar{\tau}^* \in [L_2, \tau_l + L_2)$, and as Δc increases we then see $\bar{\tau}^* \in [\tau_l + L_2, \tau_u)$. This means that in cases where the additional unit purchasing cost for using the more reliable mode is sufficiently small, it is optimal to always plan to utilize both suppliers during each replenishment cycle. As Δc increases, the use of shipping mode 2 becomes more expensive, and therefore, the model will increase the value of $\bar{\tau}^*$, such that the probability of placing and order with delivery mode 2 is strictly less than one, while decreasing the order size with mode 2.

However, even when supply mode 2 is the preferred single supplier mode, i.e., $G_2 < G_1$, and $\Delta c = 0$, we find some cases where $\bar{\tau}^*$ occurs later in the interval, i.e., $\bar{\tau}^* \in [\tau_l + L_2, \tau_u)$, as shown in Tables 2 and 3. In other words, even though mode 2 is the cheaper single mode option, it is not optimal to maximize the size of the order with supplier 2 in our dual-mode model. This occurs because when $\Delta\tau$ is sufficiently large, placing a large order with supplier 2 implies ordering early within the interval, and thus a high probability of having to place the order with supplier 2, and a correspondingly high expected cost of holding this order in inventory.

Since we do not have closed-form expressions for the optimal values of Q^* and β^* for the dual-mode model, we perform numerical tests to evaluate how changes in key model parameters for modes 1 and 2 may affect the optimal solution of the dual-mode model and its performance relative the single-mode model. Sections 3.1 through 3.8 show how changes in a single parameter of interest influence the relative performance of the dual-mode model, all else being equal. Following this, in Section 3.9 we compare the results of the dual-mode model under various combinations of mode delivery lead time and variable cost.

3.1 Impact of increasing τ_l

In order to analyze the effects of changes in the value of the lower limit on the lead time for delivery mode 1, τ_l, we performed numerical tests where we increased the value of τ_l while holding all other parameters equal.

When τ_l increases (for a fixed value of τ_u), the uncertainty in the lead time of mode 1 will decrease, and therefore, the need for an emergency supplier will also decrease. As a result, for increasing values of τ_l we expect to observe a decrease in the size of the order placed with delivery mode 2, $\lambda\beta$, and therefore, an increase in $\bar{\tau}^*$. Since an increase in τ_l represents a decrease in the value of $\Delta\tau$, we also expect to see a decrease in the average inventory cost.

Table 2 shows the optimal solutions for the dual-mode model and single-mode models and $\%CR$, for increasing values of τ_l.

As expected, we observe a decrease in the order size placed with delivery mode 2, except in the cases when $\Delta c = 0$. In this example, when $\Delta c = 0$ it is beneficial to increase the use of delivery mode 2, since we can benefit from using a more reliable mode without paying a premium in the purchasing cost, and therefore we will observe increasing values for the size of the second order, $\lambda\beta$.

We also observe a decrease in the average inventory cost per unit time, although the effect on $\%CR$ will depend on which supply mode has the minimum average inventory cost. The average cost of mode 1 decreases as we increase τ_l, since there is a reduction in the uncertainty in the arrival time of the order, and therefore, in the safety stock required. The reduction in G_1 is greater

Dual-Mode Model					Single-Mode Model				CR
τ_l days	$\bar{\tau}$ days	$\lambda\beta$ units	Q units	G^* $\$/yr$	Q_1 units	G_1 $\$/yr$	Q_2 units	G_2 $\$/yr$	%
$\Delta c = 0$									
7	22	904	1315	102,050.0	1507	102,780.1	1506	102,258.3	9.22
14	5	1233	986	101,937.3	1507	102,636.2	1506	102,258.3	14.22
20	5	1233	839	101,875.2	1507	102,513.0	1506	102,258.3	16.97
$\Delta c = 0.5$									
7	49	164	1343	102,730.0	1507	102,780.1	1506	107,258.3	1.80
14	50	137	1370	102,630.0	1507	102,636.2	1506	107,258.3	0.24
20	52	82	1425	102,540.0	1507	102,513.0	1506	107,258.3	-1.08
$\Delta c = 1$									
7	53	55	1452	102,800.0	1507	102,780.1	1506	112,258.3	-0.72
14	53	55	1452	102,680.0	1507	102,636.2	1506	112,258.3	-1.66
20	54	27	1480	102,580.0	1507	102,513.0	1506	112,258.3	-2.67

Table 2: Effect of τ_l on the dual-mode model.

than the reduction in the average inventory cost of the dual-mode model, G^*, and therefore, when $G_1 < G_2$, the value of $\%CR$ will decrease as we reduce the uncertainty in mode 1 lead time, making the dual-mode model relatively less attractive for increasing values of τ_l. Because an increase in τ_l does not affect the cost of mode 2, when $G_2 < G_1$, an increasing value of τ_l leads to an increase in $\%CR$, since G^* decreases.

3.2 Impact of increasing τ_u

When we increase τ_u while holding all other parameters equal, we increase the uncertainty of mode 1. As a result we expect to see an increase in the size of the order shipped via mode 2, since the dual-mode model will attempt to mitigate the impact of the increased uncertainty in L_1 by using shipping mode 2. Note also that an increase in the uncertainty in the arrival time of an order placed with supply mode 1 will increase the average inventory cost of using supply mode 1 exclusively, as well as the average inventory cost of the dual-mode model, although it will not affect the inventory cost associated with using supply mode 2 exclusively.

Table 3 shows the effect that increasing values of τ_u have on the dual-mode model and the single-mode model, for a specific instance.

Dual Mode-Model								Single-Mode Model				CR
τ_u^1 days	τ_u^2 days	τ_u^3 days	τ_u^4 days	$\bar{\tau}$ days	$\lambda\beta$ units	Q units	G^* $\$/yr$	Q_1 units	G_1 $\$/yr$	Q_2 units	G_2 $\$/yr$	%
$\Delta c = 0$												
30	35	40	45	5	685	1000	101,858.0	1000	102,078.8	1506	102,258.3	10.62
50	55	60	65	5	1233	986	101,937.3	1507	102,636.2	1506	102,258.3	14.22
60	65	70	75	31	932	1397	102,080.0	1781	102,945.1	1506	102,258.3	7.90
$\Delta c = 0.5$												
30	35	40	45	35	0	1247	102,300.0	1000	102,078.8	1506	107,258.3	-10.64
50	55	60	65	50	137	1370	102,630.0	1507	102,636.2	1506	107,258.3	0.24
60	65	70	75	56	247	1534	102,810.0	1781	102,945.1	1506	107,258.3	4.59
$\Delta c = 1$												
30	35	40	45	35	0	1247	102,300.0	1000	102,078.8	1506	112,258.3	-10.64
50	55	60	65	53	55	1452	102,680.0	1507	102,636.2	1506	112,258.3	-1.66
60	65	70	75	61	110	1671	102,920.0	1781	102,945.1	1506	112,258.3	0.85

Table 3: Effect of τ_u on the dual-mode model.

As expected, for increasing values of τ_u we observe an increase in the size of the order placed with mode 2 and an increase in the average inventory cost of the dual-mode model, G^*.

Note that when $\Delta c = 0$, the increase in τ_u (and therefore, increase in $\Delta\tau$) has a different effect on the quantity ordered by modes 1 and 2. As mentioned previously, when $\Delta c = 0$ and for large values of $\Delta\tau$ when that $G_2 < G_1$, the dual-mode model will try to reduce the probability of using supply mode 2 by choosing $\bar{\tau}^* \in [\tau_l + L_2, \tau_u)$, and therefore, reducing $\lambda\beta$. Also, for increasing values of $\Delta\tau$, the lower limit on the size of the order placed with supply mode 1 in order to avoid order crossing, $Q \geq \Delta\tau$ (see Appendix A) increases, and therefore, we observe that this minimum order size constraint is satisfied at equality.

We observe that the effect of increasing τ_u on $\%CR$ will depend on which supplier has the minimum average inventory cost. Since the average inventory cost of using supply mode 1 exclusively increases at a faster rate than the average inventory cost of the dual-mode model, G^*, when $G_1 < G_2$, $\%CR$ will increase. However, when $G_2 < G_1$, the percentage cost reduction will decrease, since G_2 is independent of τ_u.

3.3 Impact of increasing mode 2 lead time, L_2

We now analyze the effect of increasing the value of L_2 (while keeping the other parameters equal) on the dual-mode model. As L_2 increases, the time when we need to make the decision on whether to place the second order or not, $\bar{\tau} - L_2$, will be earlier in the cycle, and therefore, the probability of placing the second order will increase. We can view the increase in L_2 as a decrease in the responsiveness on delivery mode 2, and as a result, the average inventory cost of the dual-mode model will increase, and the model will be less beneficial.

Table 4 shows the effect of the increase in L_2 on the performance of the dual-mode model. As expected, the average inventory cost of our proposed model increases for increasing values of L_2, and since the average inventory cost of the single-mode mode is independent of L_2, we observe a decrease in $\%CR$.

Dual-Mode Model					Single-Mode Model				CR
L_2 days	$\bar{\tau}$ days	$\lambda\beta$ units	Q units	G^* $\$/yr$	Q_1 units	G_1 $\$/yr$	Q_2 units	G_2 $\$/yr$	%
$\Delta c = 0$									
5	5	1233	986	101,937.3	1507	102,636.2	1506	102,258.3	14.22
14	14	986	986	101,971.4	1507	102,636.2	1506	102,258.3	12.71
30	30	685	1123	102,181.4	1507	102,636.2	1506	102,258.3	3.40
$\Delta c = 0.5$									
5	50	137	1370	102,630.0	1507	102,636.2	1506	107,258.3	0.24
14	54	27	1480	102,790.0	1507	102,636.2	1506	107,258.3	-5.83
30	55	0	1507	102,980.0	1507	102,636.2	1506	107,258.3	-13.04
$\Delta c = 1$									
5	50	137	1452	102,680.0	1507	102,636.2	1506	112,258.3	-1.66
14	54	27	1507	102,790.0	1507	102,636.2	1506	112,258.3	-5.83
30	55	0	1507	102,980.0	1507	102,636.2	1506	112,258.3	-13.04

Table 4: Effect of L_2 on the dual-mode model.

3.4 Change in demand rate: λ

In order to evaluate how the demand rate affects the performance of the dual-mode model we performed numerical tests where we increased the value of λ, while holding other parameters equal.

As the demand rate increases we expect to observe an increase in the total quantity ordered through modes 1 and 2, i.e., $Q+\lambda\beta$, and an increase in the average inventory cost for the dual-mode model as well as for the single-mode model.

Table 5 shows the optimal solution and average inventory cost for the dual-mode model and single-mode model for increasing values of demand rate. Note that for these instances, because the demand rates used were less than the base value of 10,000, we modified the break points at which τ_u increases for increasing values of Q (see the caption of Table 5). This permitted analyzing solutions with different values of τ_u, and showing how the value of λ affects the value of τ_u in the optimal solution.

Dual-Mode Model					Single-Mode Model				CR
λ units/yr	$\bar{\tau}$ days	$\lambda\beta$ units	Q units	G^* $\$/yr$	Q_1 units	G_1 $\$/yr$	Q_2 units	G_2 $\$/yr$	%
$\Delta c = 0$									
500	50	0	270	5,442.7	258	5,424.3	337	5,505.0	-4.34
3000	5	411	527	31,042.9	500	31,196.9	825	31,237.0	12.87
7000	5	959	786	71,608.4	1055	72,044.5	1260	71,889.0	14.85
9000	5	959	1011	91,873.1	1356	92,439.0	1428	92,142.4	12.57
$\Delta c = 0.5$									
500	50	0	270	5,442.7	258	5,424.3	337	5,755.0	-4.34
3000	51	33	681	31,236.0	500	31,196.9	825	32,737.0	-3.27
7000	51	77	1053	72,079.0	1055	72,044.5	1260	75,389.0	-1.69
9000	51	77	1258	92,447.0	1356	92,439.0	1428	96,642.0	-0.33
$\Delta c = 1$									
500	50	0	270	5,442.7	258	5,424.3	337	6,005.0	-4.34
3000	55	0	659	31,241.0	500	31,196.9	825	34,237.0	-3.68
7000	54	19	1036	72,100.0	1055	72,044.5	1260	78,889.0	-2.72
9000	54	19	1332	92,488.0	1356	92,439.0	1428	101,140.0	-2.01

Table 5: Effect of demand rate on the dual-mode model.
$q^1 = 0$ *units* $q^2 = 500$ *units* $q^3 = 1,500$ *units* $q^4 = 3,000$ *units*

We observed that for increasing values of λ, when $G_1 < G_2$, the value of $\%CR$ increases. Although we have not been able to prove it analytically, the reason is that for increasing values of λ, the average inventory cost for supply mode 1 increases at a faster rate than the average inventory cost for the dual-mode model, G^*. The opposite happens when we increase λ and $G_2 < G_1$. Since G^* increases at a faster rate than the average inventory cost for supply mode 2, the value of $\%CR$ will decrease.

3.5 Change in holding cost per unit per time: h

To evaluate the effect of the holding cost per unit per unit time in the dual-mode model, we performed numerical tests where we increased the value of h and held the rest of the parameters fixed.

Table 6 shows the order quantities and average inventory cost for the single-mode model and the dual-mode model. We observe that when h increases, the total quantity ordered in the dual-mode model, $Q + \lambda\beta$, decreases and the average inventory cost increases.

Through our numerical tests, we cannot predict the effects of the increase of h on $\%CR$, since it will depend on the relative value of h with respect to the other parameters.

Dual-Mode Model					Single-Mode Model				CR
h \$/unit · yr	$\bar{\tau}$ days	$\lambda\beta$ units	Q units	G^* $\$/yr$	Q_1 units	G_1 $\$/yr$	Q_2 units	G_2 $\$/yr$	%
$\Delta c = 0$									
0.8	5	1370	1173	101,387.4	1507	101,715.7	2062	101,649.2	15.88
1.2	5	1233	986	101,703.0	1507	102,241.7	1683	102,019.9	15.69
2.5	26	658	986	102,660.0	1507	103,951.3	1166	102,915.5	8.76
$\Delta c = 0.5$									
0.8	54	27	1670	101,770.0	1507	101,715.7	2062	106,649.2	-3.17
1.2	52	82	1425	102,280.0	1507	102,241.7	1683	107,019.9	-1.71
2.5	45	274	1233	103,650.0	1507	103,951.3	1166	107,915.5	7.63
$\Delta c = 1$									
0.8	55	0	1647	101,770.0	1507	101,715.7	2062	111,649.2	-3.17
1.2	55	0	1507	102,300.0	1507	102,241.7	1683	112,019.9	-2.60
2.5	50	137	1370	103,860.0	1507	103,951.3	1166	112,915.5	2.31

Table 6: Effect of holding cost on the dual-mode model.

3.6 Change in incremental fixed order cost for mode 2: $\bar{A}$

When the incremental fixed order cost for using shipping mode 2 in the dual-mode model increases, while holding all other parameters fixed, we expect to observe an increase in the average inventory cost of the dual-mode model, since it is more expensive to place a second order.

Since the average inventory cost for the single-mode model is independent of the extra fixed order cost of placing a second order, as we increase $\bar{A}$ the value of $\%CR$ will decrease, making the dual-mode model less beneficial.

Table 7 shows the results of our numerical tests when we increase the value of $\bar{A}$.

Dual Mode Model					Single Mode Model				CR
$\bar{A}$ \$/order	$\bar{\tau}$ days	$\lambda * \beta$ units	Q units	G^* $\$/yr$	Q_1 units	G_1 $\$/yr$	Q_2 units	G_2 $\$/yr$	%
$\Delta c = 0$									
70	5	1233	986	101,937.3	1507	102,636.2	1506	102,258.3	14.22
100	5	1233	986	102,072.5	1507	102,636.2	1506	102,258.3	8.23
150	5	1233	1000	102,297.4	1507	102,636.2	1506	102,258.3	-1.73
$\Delta c = 0.5$									
70	50	137	1370	102,630.0	1507	102,636.2	1506	107,258.3	0.24
100	51	110	1397	102,680.0	1507	102,636.2	1506	107,258.3	-1.66
150	53	55	1452	102,740.0	1507	102,636.2	1506	107,258.3	-3.94
$\Delta c = 1$									
70	53	55	1452	102,680.0	1507	102,636.2	1506	112,258.3	-1.66
100	54	27	1480	102,710.0	1507	102,636.2	1506	112,258.3	-2.80
150	55	0	1507	102,760.0	1507	102,636.2	1506	112,258.3	-4.69

Table 7: Effect of $\bar{A}$ on the dual-mode model.

3.7 Change in purchasing cost when using shipping mode 2: c_2

When we increase the purchasing cost per unit for supply mode 2, and keep all other parameters fixed, we expect to observe an increase in the average inventory cost of the dual-mode model. Tables

2 to 7 show the optimal solution for the dual-mode model and single-mode model for different values of c_2.

We observe that although it may be intuitive to conclude that as we increase c_2 the use of the dual-mode model will become less attractive, that is not always the case. The impact of the change in c_2 depends on which shipping mode has the minimum cost when they are used independently. When shipping mode 2 is the preferred supply mode, as we increase c_2, the cost of using it exclusively also increases, but at a faster rate than the increase in the average inventory cost of the dual-mode model, G^*, and therefore the percentage cost reduction obtained by using the dual-mode model increases with increasing values of c_2.

This increasing trend in $\%CR$ stops when c_2 is large enough to make mode 1 the new preferred shipping mode. When mode 1 is the preferred single supply mode, as we increase c_2, the value of G_1 does not change, and $\%CR$ decreases. Therefore the maximum $\%CR$ for increasing values of c_2 is reached at the maximum value of c_2 such that $G_2 \leq G_1$.

3.8 Change in the lead team distribution for supply mode 1

For the development of our model we assumed that the lead time for supply mode 1 is Uniformly distributed. When the mode 1 lead time follows a Uniform Distribution, $L_1 \sim U(\tau_l, \tau_u)$, the probability density of an order arrival at time t is equal for any time $t \in [\tau_l, \tau_u]$, and the expected arrival time is $E[L_1] = \frac{\tau_l + \tau_u}{2}$, which is equal to the midpoint of the interval.

We are interested in the effects that using a non-Uniform distribution for the mode 1 lead time has on the dual-mode model. In particular, we assume that $L_1 \sim \{\tau_l + \Delta\tau Beta(a, b)\}$[1] where $a = 2$ and $b > 2$, and therefore the distribution is unimodal and positively skewed. Under this assumption the probability of having an early arrival (before the middle point of the interval) is higher than when a Uniformly distributed lead time is assumed.

Since we can no longer use the expressions found in Section 2, in order to find the optimal solution for the dual-mode model when we use a Beta distribution for L_1, we need to use a simulation-based optimization approach; in particular, we used the commercial package OptQuest from Arena.

Table 8 shows the optimal values and average inventory costs for the single-mode model and dual-mode model when we assume that $L_1 \sim \{\tau_l + \Delta\tau Beta(a, b)\}$. The last two columns of Table 8 show the $\%CR$ values for the different assumptions for the distribution of L_1.

We observed that we obtained higher $\%CR$ values when we assume that L_1 follows a Beta distribution, and the reasons for this are twofold: first, the average inventory cost of using supply mode 1 exclusively is higher when we use a positive skewed Beta distribution for L_1, compared with the average inventory cost of assuming a Uniformly distributed lead time. This is because the expected arrival time when $L_1 \sim \{\tau_l + \Delta\tau Beta(a, b)\}$ is less than $\frac{\tau_l + \tau_2}{2}$, and therefore, the holding cost due to safety stock $(\lambda\tau_u - \lambda E[L_1])$ is increased under the Beta distribution (when compared to the Uniform); second, since the probability of an early arrival is higher when we assume that L_1 follows a Beta distribution, compared with the use of the Uniform distribution, the probability of placing the second order with supply mode 2 decreases, and therefore, the average total cost of using the dual-mode model will decrease.

In practice we expect that L_1 follows a probability distribution with some positive skew, since the unusually long cases typically lead to the skew in the distribution. Hence, based on the previous analysis, we can expect to have higher $\%CR$ values than the ones observed under the assumption of a Uniformly distributed lead time, and therefore, the dual-mode model may be more beneficial in practice.

[1] We define L_1 as a scaled Beta variable, since the Beta distribution is defined in the interval $[0, 1]$ and we are interested in general cases where $L_1 \in [\tau_l, \tau_u]$.

Dual-Mode Model					Single-Mode Model				CR	
λ units/yr	$\bar{\tau}$ days	$\lambda\beta$ units	Q units	G^* $\$/yr$	Q_1 units	G_1 $\$/yr$	Q_2 units	G_2 $\$/yr$	% $L_1 \sim \{\tau_l + \Delta\tau Beta\,(a,b)\}$	% $L_1 \sim U(\tau_l, \tau_u)$
$\Delta c = 0$										
500	45	7	241	5433.6	258	5,440.1	337	5505.0	1.51	-4.34
3000	5	411	585	30987.9	500	31,292.0	825	31237.0	20.14	12.87
7000	7	921	604	71450.6	1055	72,297.2	1260	71889.0	23.23	14.85
$\Delta c = 0.5$										
500	43	10	238	5428.3	258	5,440.1	337	5755.0	2.69	-4.34
3000	43	99	669	31196.1	500	31,292.0	825	32737.0	7.42	-3.27
7000	42	249	197	71980.2	1055	72,297.2	1260	75389.0	13.80	-1.69
$\Delta c = 1$										
500	43	10	241	5430.9	258	5,440.1	337	6005.0	2.10	-4.34
3000	41	74	500	31181.7	500	31,292.0	825	34237.0	8.54	-3.68
7000	42	74	1013	72000.9	1055	72,297.2	1260	78889.0	12.90	-2.72

Table 8: Dual-mode model for $L_1 \sim \{\tau_l + \Delta\tau Beta\,(a,b)\}$.
$q^1 = 0$ *units* $q^2 = 500$ *units* $q^3 = 1,500$ *units* $q^4 = 3,000$ *units*

3.9 Comparing supply modes with different cost and lead time parameters

The analysis discussed in the previous sections shows how certain parameter changes affect the benefits of the dual-mode model. This section considers how a buyer might evaluate multiple supply mode options, where a given mode implies certain costs and lead time distribution characteristics. For example, using the example in Figure 10, we can see how the buyer might compare different perfectly reliable suppliers with different combinations of purchasing costs and lead times.

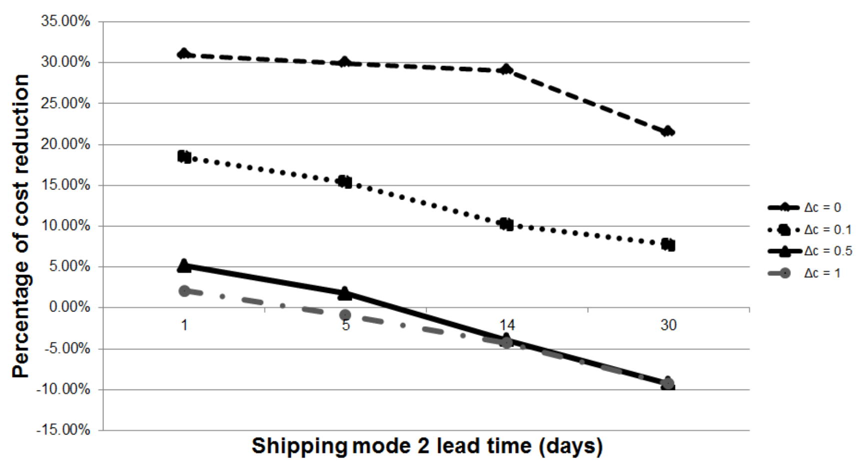

Figure 10: Percentage of cost reduction for different values of L_2 and Δc.
$A_2 = \$150/order$ $\bar{A} = \$50/order$

In Figure 10 we observe that choosing the supply mode with the shortest lead time, while keeping the other parameters equal, will provide the biggest $\%CR$ value. However, in practice, a reduction in the lead time corresponds to an increase in the unit purchasing cost. As a result it may be more beneficial to choose a shipping mode with a longer lead time but with a reduced unit purchasing cost. In this way the buyer can negotiate purchasing costs based on delivery times.

Next we would like to compare different supply modes with different uncertainty levels. The curves in Figure 11 represent different shipping modes with Uniformly distributed lead times, each with the same mean, but with different values of variance (where $\Delta\tau = 41$ days for the base case). This means that we will compare shipping modes with equal values for the expected arrival time, but with different values of $\Delta\tau$.

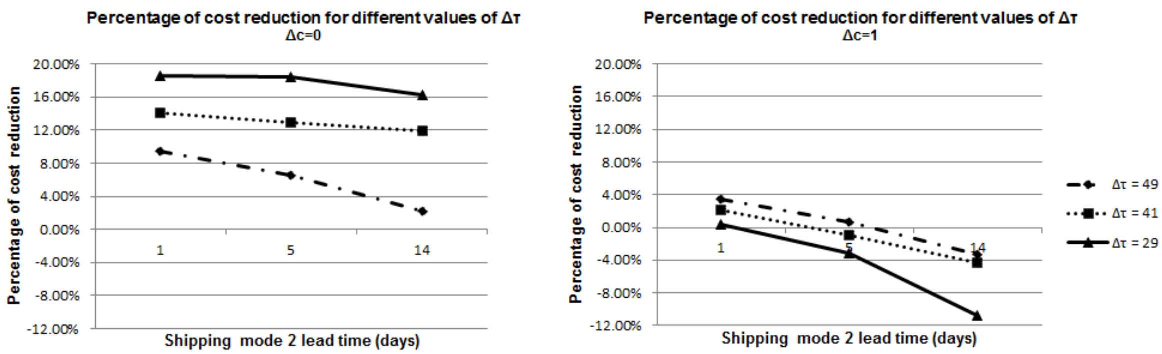

Figure 11: Percentage of cost reduction for different values of L_2, Δc and $\Delta\tau$.
$A_2 = \$150/order\ \bar{A} = \$50/order$

Although intuition suggests that the dual-mode model will be more beneficial as the uncertainty in the lead time of shipping mode 1 increases, as shown in Figure 11, when $\Delta c = 0$, this is not always the case. When $\Delta c = 0$ we obtain the greatest benefit when $\Delta\tau = 29$ days. The reason for this is that, for this instance, when $\Delta c = 0$, mode 2 is the preferred supplier ($G_2 < G_1$), and therefore, as we decrease $\Delta\tau$ the average inventory cost for supply mode 2 does not change, while the average inventory cost of the dual-mode model decreases, and consequently $\%CR$ increases.

Next we consider the dual-mode model for different modes with Uniformly distributed lead times and equal uncertainty levels, but increasing values of the expected delivery time. This means that we are comparing shipping modes with equal values of $\Delta\tau$, but increasing values of $E[L_1]$, while holding all other parameters equal. Figure 12 shows the results, where $E[L_1] = 34.5$ days for the base case.

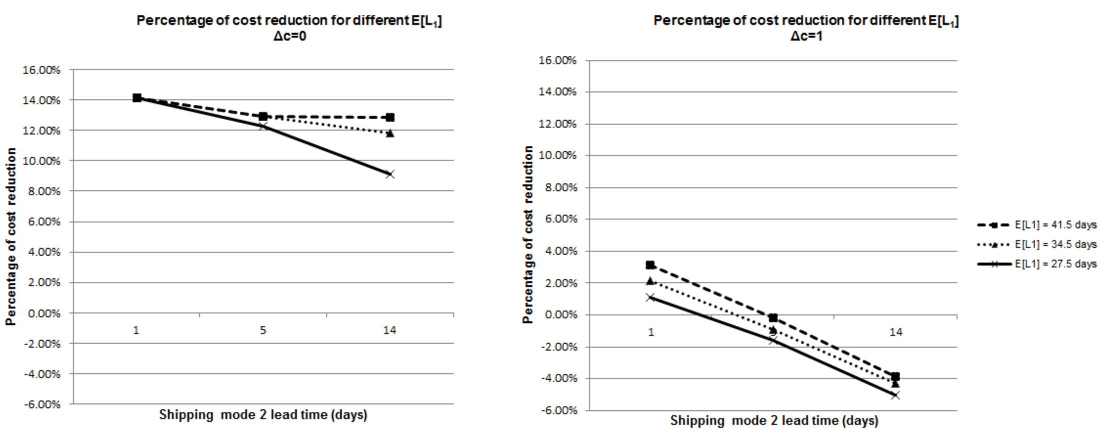

Figure 12: Percentage of cost reduction for different values of L_2, Δc and $\hat{L}_1$.
$A_2 = \$150/order\ \bar{A} = \$50/order$

Note that the average inventory cost of using mode 2 exclusively is independent of $E[L_1]$; therefore for instances with different expected lead times for supply mode 1, the value of G_2 will not change.

For the single-mode model when we use mode 1 exclusively, in order to avoid stockouts we need an order size at least as great as the reorder point, i.e., $Q \geq \lambda\tau_u$; therefore for increasing values of $E[L_1]$, the right hand side of the inequality increases and we will have increasing values of G_1.

As expected, we have increasing values of $\%CR$ for increasing $E[L_1]$. Note that when $\Delta c = 0$ we have some cases when the $\%CR$ values are equal for for different $E[L_1]$. There are two reasons for this results: first, when $\Delta c = 0$ supply mode 2 is chosen as regular supplier ($G_2 < G_1$), and therefore, we will compare the dual-mode model against the same value for the average inventory cost for the single-mode model, for different values of $E[L_1]$; and second, for small values of L_2 the constraint on order size to avoid order crossing, $Q \geq \lambda\Delta\tau$, is more restrictive than $Q \geq \lambda\bar{\tau}$, and therefore for different values of $E[L_1]$ and equal values of $\Delta\tau$, the values of Q and $\lambda\beta$ are equal for different instances, which results in the same average inventory cost for the dual-mode model.

As we can see, the proposed model presents saving opportunities for the buyer, and it can be used in a joint manner with the supplier to achieve inventory cost reductions for the buyer and better resource allocation for the supplier.

4 Conclusions, Recommendations, and Suggested Research

When allocating orders to suppliers, there is often a tradeoff between cost and delivery performance. The use of a cheaper supply mode often implies dealing with higher uncertainty levels when compared with a more expensive mode. Assuming that shortages are highly undesirable, if a cheap and less reliable delivery mode is used, it may be beneficial to consider supplementing this with a secondary supply mode that is more reliable (and therefore more expensive). This paper focused on the benefits of this dual-mode ordering strategy by modeling a buyer's ordering decisions with two delivery mode options. Our model allowed us to characterize the benefits of the dual-mode supply strategy and compare the optimal dual-mode policy with the better of the two single-supplier solutions.

Using numerical analysis, we characterized situations under which the use of the dual-mode model is preferred over the single-mode model. We observed increasing benefits in our model for increasing values in the mean and variance in the lead time of the less reliable supply mode (assuming that supply mode 1 is used as the preferred single-mode supplier). We also observed that when the supply mode 1 lead time follows a positively skewed probability distribution, which is typically a more realistic assumption than a Uniform lead time, the percentage cost reduction due to the dual-mode operation is greater compared with the results when we assume that supply mode 1 lead time is Uniformly distributed.

Our model can be used by a manager who wants to analyze different combinations of shipping modes in order to minimize inventory costs while guaranteeing zero stockouts. Our work shows that a supplier that offers different shipping mode options with different delivery costs and reliability levels may provide additional value to potential buyers. Further research may consider different supplier pricing and incentives. For example, in order to increase the utilization of shipping modes with available capacity, a supplier might offer quantity discounts for the use of those modes, making shipping diversification more attractive for the buyer. Finally, we note that one dimension our model does not account for is the risk preference of the decision maker, effectively assuming that the buyer is risk neutral. An interesting extension of our work may include accounting for the way in which different buyer risk profiles influence the allocation of order quantities to different supply modes.

Appendices

A Order crossing condition

Assuming that we use a supply mode with a stochastic lead time, $L_1 \sim U(\tau_l, \tau_u)$ and a constant demand rate λ, if we use a continuous review (Q, r) policy, where $r = \lambda\tau_u$, we need $Q \geq \lambda(\tau_u - \tau_l)$ to prevent order crossing. In order to prove this assertion, we assume for a contradiction that $Q < \lambda(\tau_u - \tau_l)$.

Note that the inventory position at any time $t \geq 0$ is equal to $Q + \lambda\tau_u - \lambda t$. When the inventory position reaches the reorder point $r = \lambda\tau_u$, an order is placed. That is, when $Q + \lambda\tau_u - \lambda t = \lambda\tau_u$, an order is placed. This implies that an order is placed at time $t = \frac{Q}{\lambda}$ and by the assumption that $Q < \lambda(\tau_u - \tau_l)$, we have that:

$$t = \frac{Q}{\lambda} < \tau_u - \tau_l.$$

An order placed at time t will arrive at time $t + L_1$, and with probability strictly greater than zero, the order will arrive at time $t + \tau_l$, and since $t < \tau_u - \tau_l$, we have that:

$$t + \tau_l < \tau_u - \tau_l + \tau_l = \tau_u$$

Therefore, with probability strictly greater than zero the order may be received before τ_u, which implies a positive probability of order crossing. Thus an absence of order crossing implies $Q \geq \lambda(\tau_u - \tau_l)$.

B Single-Mode Solution with Quantity Based Upper Bound

This section presents the solution procedure to find an optimal order quantity from mode 1, Q_1^*, and minimum average inventory cost, G_1^* for the single-mode model, when the lead time upper bound for supply mode 1, τ_u, is a step function of the order size Q such that, $\tau_u = \tau_u^i$ if $q_i \leq Q < q_{i+i}$ for $i = 1, 2, ...n$ where $q_i < q_{i+1}$, $\tau_u^i < \tau_u^{i+1}$ and n is the number of intervals.

1. Find Q_1^* using (2.2);
2. Find $G_1^k(Q_1^*)$, for $\tau_u = \tau_u^k$ such that $q_k < Q_1^* \leq q_{k+1}$, using (2.3). Set $G_1^* = G_1^k(Q^*)$;
3. **while** $k > 1$ **do**
4. Find $G_1^{k-1}\left(q_k, \tau_u^{k-1}\right)$ using (2.1), which is the average inventory cost per unit time using the right break point for Q in the $(k-1)^{th}$ interval;
5. **if** $G_1^* \leq G_1^{k-1}$ **then**
6. Q_1^* is the optimal order quantity and the minimum average inventory cost is G_1^*. Set $k = 1$;
7. **else**
8. Set $Q_1^* = q_k$, $G_1^* = G_1^{k-1}$, $k = k - 1$;
9. **end if**
10. **end while**

C Convexity conditions for $G_I(Q, \beta)$

This section presents convexity conditions for the average inventory cost function for the dual-mode model when $\bar{\tau} \in [L_2, L_2 + \tau_l)$, represented by:

$$G_I(Q,\beta) = \frac{\lambda^2\beta^2 h + 2c_2\lambda^2\beta + hQ^2 + 2c_1\lambda Q + h\lambda Q(\tau_u - \tau_l) + 2\lambda A}{2(Q+\lambda\beta)}$$

First, we analyze average inventory cost as a function of Q and β independently, and then, we will present the convexity condition for G_I as a joint function of Q and β.

To analyze the average inventory cost as a function of Q we need to find its first and second derivatives:

$$\frac{\partial G_I}{\partial Q} = \frac{hQ^2 + 2\lambda hQ\beta + h\lambda^2\beta\Delta\tau - 2\lambda^2\beta\Delta c - \lambda^2 h\beta^2 - 2\lambda A}{2(Q+\lambda\beta)^2}$$
$$\frac{\partial^2 G_I}{\partial Q^2} = \frac{2\lambda^2 h\beta^2 - \lambda^2 h\beta\Delta\tau + 2\lambda^2\beta\Delta c + 2\lambda A}{(Q+\lambda\beta)^3}$$

Since Q and β are strictly positive, the denominator of $\frac{\partial^2 G_I}{\partial Q^2}$ is strictly positive as well. The numerator can be analyzed as a quadratic function of β of the form $h(\beta) = a\beta^2 + b\beta + c$, where $a = 2h\lambda^2$, $b = 2\lambda^2\Delta c - h\lambda^2\Delta\tau$ and $c = 2\lambda A$.

Since $a > 0$, $h(\beta)$ is convex, and the values of $h(\beta)$ will depend on its discriminant $\delta = b^2 - 4ac$. In particular, we are interested in the conditions under which $h(\beta) \geq 0$, since we need $\frac{\partial^2 G_I}{\partial Q^2} \geq 0$ for $G_I(Q)$ to be convex. Note, that when $\delta \leq 0$, $h(\beta)$ has at most one real root, and therefore, $h(\beta) \geq 0 \ \forall \beta$, and when $\delta > 0$, $h(\beta)$ will have two real roots, β_1 and β_2, and $h(\beta)$ will be positive for $\beta \in (-\infty, \beta_1] \bigcup [\beta_2, \infty)$, and negative otherwise.

In order to have $\delta \leq 0$ we need:

$$\omega^2 \leq 4\lambda hA, \tag{C.1}$$

where $\omega = h\frac{\lambda\Delta\tau}{2} - \lambda\Delta c$. This condition requires that the square of the upper bound on the difference between the average inventory cost of using shipping mode 1 and the average inventory cost of using shipping mode 2, must be less than or equal to twice the square of the ordering and holding cost of the EOQ model with holding cost per unit per unit time h and ordering cost $A = A_1 + \bar{A}$. Therefore, when condition (C.1) holds, we have that $\partial^2 f/\partial Q^2$ is non-negative for $Q \geq 0$ and $\beta \geq 0$, and hence, G_I is convex for $Q \geq 0$ and $\beta \geq 0$.

If condition (C.1) does not hold, we have that $\delta > 0$; therefore $h(\beta)$ will have two real roots: $\beta_1 = \frac{\omega - \sqrt{(-\omega)^2 - 4\lambda hA}}{2\lambda h}$ and $\beta_2 = \frac{\omega + \sqrt{(-\omega)^2 - 4\lambda hA}}{2\lambda h}$, and $h(\beta)$ will be positive for $\beta \in (-\infty, \beta_1] \bigcup [\beta_2, \infty)$.

We can see that the sign of β_1 and β_2 depends on the value of ω. When $\omega < 0$, we have $\beta_1 < \beta_2 < 0$, which means that the two real roots of $h(\beta)$ are negative, and therefore $h(\beta) > 0$ $\forall \beta \geq 0$, and consequently $\frac{\partial^2 f}{\partial \beta^2} > 0$. If $\omega > 0$, we have that $\beta_2 > \beta_1 > 0$, and therefore, $\frac{\partial^2 f}{\partial \beta^2} \geq 0$ when $\beta \in [0, \beta_1] \bigcup [\beta_2, \infty)$.

Finally we can conclude that $G_I(Q)$ is convex $\forall Q > 0$ and $\beta > 0$ when (C.1) holds or when $\omega < 0$. If (C.1) does not hold and $\omega > 0$, then $G_I(Q)$ is convex $\forall Q > 0$ and $\beta \in [0, \beta_1] \bigcup [\beta_2, \infty)$.

We next analyze the average inventory cost as function of β using its first and second derivatives:

$$\frac{\partial G_I}{\partial \beta} = \frac{2\lambda^2 h\beta Q + 2\lambda^2 Q\Delta c + \lambda^3\beta^2 h - \lambda hQ^2 - \lambda^2 hQ\Delta\tau - 2\lambda^2 A}{2(Q+\lambda\beta)^2}$$
$$\frac{\partial^2 G_I}{\partial \beta^2} = \frac{2\lambda^2 hQ^2 - 2\lambda^3 Q\Delta c + \lambda^3 hQ\Delta\tau + 2\lambda^3 A}{(Q+\lambda\beta)^3}$$

We observe that the denominator of $\frac{\partial^2 G_I}{\partial \beta^2}$ is positive $\forall Q > 0$ and $\beta > 0$ and that the numerator is a quadratic function of Q of the form $k(Q) = aQ^2 + bQ + c$, where $a = 2\lambda^2 h$, $b = h\lambda^3\Delta\tau - 2\lambda^3\Delta c$ and $c = 2\lambda^3 A$ with roots: $Q_1 = \frac{-\omega - \sqrt{\omega^2 - 4\lambda A h}}{2h}$ and $Q_2 = \frac{-\omega + \sqrt{\omega^2 - 4\lambda A h}}{2h}$

Following the same analysis used for $G_I(Q)$, we have that $G_I(\beta)$ is convex $\forall Q > 0$ and $\beta > 0$ if condition (C.1) holds or if $\omega > 0$. If condition (C.1) does not hold and $\omega < 0$, $G_I(\beta)$ is convex for $Q \in [0, Q_1] \bigcup [Q_2, \infty)$ and $\beta > 0$.

Note that (C.1) it is not restrictive, since the order cost is typically bigger than Δc and than the holding cost per unit during $\Delta\tau$. For the rest of this section we will assume that (C.1) holds, and therefore $G_I(\beta)$ and $G_I(Q)$ are convex $\forall \beta > 0$ and $Q > 0$.

To establish convexity conditions for G_I as a joint function of Q and β we need to analyze its Hessian:

$$H = \begin{bmatrix} \frac{\partial^2 G_I}{\partial Q^2} & \frac{\partial^2 G_I}{\partial Q \partial \beta} \\ \frac{\partial^2 G_I}{\partial \beta \partial Q} & \frac{\partial^2 G_I}{\partial \beta^2} \end{bmatrix},$$

where:

$$\frac{\partial^2 G_I}{\partial \beta \partial Q} = \frac{\lambda\omega(Q - \lambda\beta) - 2\lambda^2 h Q\beta + 2\lambda^2 A}{(Q + \lambda\beta)^3}$$

Since $\frac{\partial^2 G_I}{\partial \beta^2}$ and $\frac{\partial^2 G_I}{\partial Q^2}$ are non-negative when condition (C.1) holds, we need to transform H to the form:

$$H^{new} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & h_{22}^{new} \end{bmatrix}$$

In order to do the transformation we use $F = -\frac{\lambda\omega(Q-\lambda\beta) - 2\lambda^2 hQ\beta + 2\lambda^2 A}{2\lambda^2 h\beta^2 - 2\lambda\omega\beta + 2\lambda A}$, and we have that,

$$\begin{aligned} h_{22}^{new} &= Fh_{12} + h_{22} \\ &= \frac{\lambda^2\left(4\lambda A h - \omega^2\right)}{2\lambda^2 h\beta^2 - 2\lambda\omega\beta + 2\lambda A} \end{aligned}$$

Substituting $h(\beta) = 2\lambda h\beta^2 - 2\omega\beta + 2A$, we have:

$$h_{22}^{new} = \frac{\lambda^2\left(4\lambda A h - \omega^2\right)}{h(\beta)}.$$

Because we assume that (C.1) holds, $h_{22}^{new} \geq 0$, and this implies that H is positive semidefinite, and therefore $G_I(Q, \beta)$ is convex $\forall Q > 0$ and $\beta > 0$.

D Algorithm for dual-mode model when $\bar{\tau} \in [L_2, \tau_1 + L_2)$

Algorithm 1 shows a detailed description of the solution procedure to find the optimal order quantities from modes 1 and 2, and minimum average inventory cost for the dual-mode model when the lead time upper bound of mode 1, τ_u, is a step function of Q and $\bar{\tau} \in [L_2, \tau_1 + L_2)$. Note that Algorithm 1 uses the expressions described in Section 2.2.1.

E Algorithm for dual-mode model when $\bar{\tau} \in [\tau_1 + L_2, \tau_u)$

Algorithm 2 shows a detailed description of the solution procedure to find the optimal order quantities from modes 1 and 2, and minimum average inventory cost for the dual-mode model when the lead time upper bound of mode 1, τ_u, is a step function of Q and $\bar{\tau} \in [\tau_1 + L_2, \tau_1 + L_2]$. Note that Algorithm 2 uses the expressions described in Section 2.2.2.

Algorithm 1 Minimum average inventory cost for the dual-mode model when $\bar{\tau} \in [L_2, \tau_1 + L_2)$.

$i \leftarrow 1$
while $i \leq n$ **do**
  Find β_i^* and Q_i^* for τ_u^i
  if $Q_i^* \in (q_i, q_{i+1}]$ **then**
    Find the minimum average inventory cost for the i^{th} interval G_I^i
  else if $Q_i^* < q_i$ **then**
    if $G_I^i(q_i, \beta(q_i)) \leq G_I^{i-1}(q_i, \beta(q_i))$ **then**
      The minimum average inventory cost for the i^{th} interval is $G_I^i(q_i, \beta(q_i))$
    else
      The solution is not in the i^{th} interval
    end if
  else $Q_i^* > q_{i+1}$
    if $G_I^i(q_{i+1}, \beta(q_{i+1})) \leq G_I^{i+1}(q_{i+1}, \beta(q_{i+1}))$ **then**
      The minimum average inventory cost for the i^{th} interval is $G_I^i(q_{i+1}, \beta(q_{i+1}))$
    else
      The solution is not in the i^{th} interval
    end if
  end if
end while
$G_I^* = \min\left\{G_I^i \; \forall i \in [1, n]\right\}$

F Validity of average inventory expression for dual-mode model when $\bar{\tau} \in [\tau_l + L_2, \tau_u]$

We consider our inventory model as a Renewal Reward Process, where the interarrival times T_n, $n \geq 1$, are equal to the length of the replenishment cycle, and each time a renewal occurs we receive a reward I_n, $n \geq 1$, equal to the average inventory in the cycle.

We observe that the pair (T_n, I_n), $n \geq 1$ are independent and identically distributed and that $E[T_n] = E[T]$ from (2.13), and $E[I_n] = E[I]$ from (2.14). Using the result showed by Ross [15], we have a Renewal Reward Process where $E[T] < \infty$ and $E[I] < \infty$, and therefore, with probability 1,

$$\frac{E[I(t)]}{t} \to \frac{E[I]}{E[T]} \text{ as } t \to \infty.$$

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Algorithm 2 Minimum average inventory cost for the dual-mode model when $\bar{\tau} \in [\tau_1 + L_2, \tau_1 + L_2]$.

$i \leftarrow 1$
while $i \leq n$ **do**
    Find β_i^* and Q_i^* for τ_u^i
    if $Q_i^* \in (q_i, q_{i+1}]$ **then**
        Find the minimum average inventory cost for the i^{th} interval G_{II}^i
    else if $Q_i^* < q_i$ **then**
        if $G_{II}^i (q_i, \beta(q_i)) \leq G_{II}^{i-1} (q_i, \beta(q_i))$ **then**
            The minimum average inventory cost for the i^{th} interval is $G_{II}^i (q_i, \beta(q_i))$
        else
            The solution is not in the i^{th} interval
        end if
    else $Q_i^* > q_{i+1}$
        if $G_{II}^i (q_{i+1}, \beta(q_{i+1})) \leq G_{II}^{i+1} (q_{i+1}, \beta(q_{i+1}))$ **then**
            The minimum average inventory cost for the i^{th} interval is $G_{II}^i (q_{i+1}, \beta(q_{i+1}))$
        else
            The solution is not in the i^{th} interval
        end if
    end if
end while
$G_{II}^* = \min \left\{ G_{II}^i \ \forall i \in [1, n] \right\}$

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