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REFERENCE USE ONLY

MODELS OF  
RAILROAD PASSENGER-CAR REQUIREMENTS  
IN THE NORTHEAST CORRIDOR

VOLUME I: FORMULATION AND RESULTS

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FINAL REPORT

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16. Abstract  Models and techniques for determining passenger-car requirements in railroad service were developed and applied by a research project of which this is the final report. The report is published in two volumes, as follows:  <u>Volume I: Formulation and Results.</u> The first part of this volume considers a general problem of determining optimal passenger-car allocations given a fixed schedule and predetermined demands. Requirements for car movements are modeled as a set of linear constraints having a transshipment structure, and alternative linear objectives are formulated. Various optimization techniques are developed for one or more objectives, and properties of the sets of optimal solutions are demonstrated.  The remainder of Volume I shows how the linear model and optimization techniques may be applied to the Northeast Corridor. Derivations of a schedule and demands are explained, and results of a number of optimizations and analyses are displayed.  <u>Volume II: User's Guide.</u> The solution and analysis of the Northeast Corridor models required the creation of a number of computer programs of several kinds. These programs are available for the use of others and are described in Volume II of this report.			
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## PREFACE

In 1973, Congress passed the Regional Railroad Reorganization Act which became law on January 2, 1974. This complex piece of legislation dealing with passenger as well as freight operations called for the Department of Transportation (DOT) to implement improved passenger rail service in the Northeast Corridor (NEC) as recommended in the 1971 Northeast Corridor Report. Planning for the improved service included engineering studies, demand projections, and financial analysis. This study uses the demand projections to compute fleet requirements for the Corridor. The results contained in the base runs do not represent official or final estimates of fleet size, train schedules, and fleet management but are illustrative examples based upon assumptions stated in the text.

The work described herein was performed for the Transportation Systems Center under contract no. DOT-TSC-1179 with the National Bureau of Economic Research. The contract was carried out by Robert Fourer at NBER's Computer Research Center for Economics and Management Science. Judith Gertler and Howard Simkowitz of TSC developed the demand-projection procedures and provided general guidance for the project. The PL/I routines for simulating annual patronage were written by John Prokopy and Diane Ruina of Peat, Marwick, Mitchell, and Co.

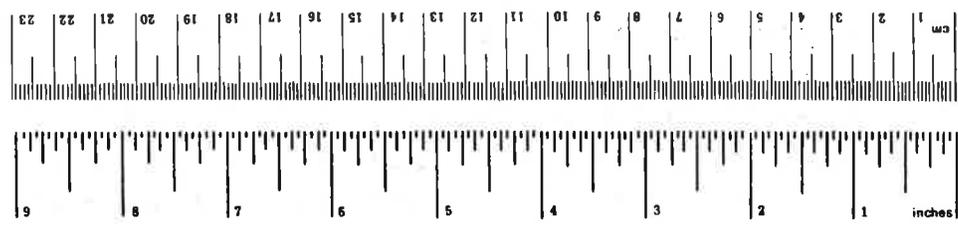
The author wishes to thank William Northup and Michael Harrison of NBER for their occasional technical assistance, and Karen Glennon for her patience in typing the manuscript.

Prospective users of the programs documented in Volume II should contact the author, care of NBER, or Howard Simkowitz at the Transportation Systems Center.

# METRIC CONVERSION FACTORS

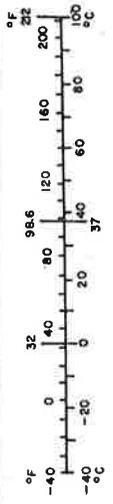
## Approximate Conversions to Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
<b>LENGTH</b>				
in	inches	2.5	centimeters	cm
ft	feet	30	meters	m
yd	yards	0.9	kilometers	km
mi	miles	1.6		
<b>AREA</b>				
in <sup>2</sup>	square inches	6.5	square centimeters	cm <sup>2</sup>
ft <sup>2</sup>	square feet	0.09	square meters	m <sup>2</sup>
yd <sup>2</sup>	square yards	0.8	square meters	m <sup>2</sup>
mi <sup>2</sup>	square miles	2.6	square kilometers	km <sup>2</sup>
	acres	0.4	hectares	ha
<b>MASS (weight)</b>				
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2000 lb)	0.9	tonnes	t
<b>VOLUME</b>				
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounce	30	milliliters	ml
c	cups	0.24	liters	l
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft <sup>3</sup>	cubic feet	0.03	cubic meters	m <sup>3</sup>
yd <sup>3</sup>	cubic yards	0.76	cubic meters	m <sup>3</sup>
<b>TEMPERATURE (exact)</b>				
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C



## Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
<b>LENGTH</b>				
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
km	kilometers	0.6	miles	mi
<b>AREA</b>				
cm <sup>2</sup>	square centimeters	0.16	square inches	in <sup>2</sup>
m <sup>2</sup>	square meters	1.2	square yards	yd <sup>2</sup>
km <sup>2</sup>	square kilometers	0.4	square miles	mi <sup>2</sup>
ha	hectares (10,000 m <sup>2</sup> )	2.5	acres	ac
<b>MASS (weight)</b>				
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1000 kg)	1.1	short tons	st
<b>VOLUME</b>				
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
m <sup>3</sup>	cubic meters	0.26	gallons	gal
m <sup>3</sup>	cubic meters	35	cubic feet	ft <sup>3</sup>
m <sup>3</sup>	cubic meters	1.3	cubic yards	yd <sup>3</sup>
<b>TEMPERATURE (exact)</b>				
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



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PART A

A GENERAL NETWORK MODEL  
FOR RAILROAD PASSENGER-CAR ALLOCATION

The study described in this report began as an attempt to determine the minimum number of passenger cars required to serve the Northeast Corridor.

The desired analyses of rail service require a mathematical model of passenger-car movements, and techniques for applying that model to produce usable data. This study employs an optimization model that has a network form, and that is amenable to linear-programming methods. A description and analysis of this model constitutes the first part of this report.

An initial statement of the problem constraints is given in §1. These constraints are subsequently modeled in network and linear-programming forms, and are augmented to represent several extensions of the problem.

Objective functions for the model are the topic of §2. Several linear objectives proportional to costs are proposed, and techniques for optimizing more than one objective are developed.

The contents of §1 and §2, though general and theoretical, are the basis for the Northeast Corridor analyses in Part B of this report. Readers interested primarily in the applications should find it sufficient to skim Part A.



## §1 FORMULATION OF THE MODEL

Fundamentally, the aim of this study is to allocate passenger cars in a railroad system. This section specifies the nature of such a system and the requirements that must be met by any feasible allocation of cars.

An informal statement of the problem occupies §1.1. The constraints are then formulated more precisely, first as a transshipment network (§§1.2-1.3), then as a linear program (§1.4) to which the simplex method may be applied.

The remainder of the section is concerned with extensions of the original problem, to model corridor service with turnaround delays (§1.5), upper limits on train sizes (§1.6), and locomotive requirements (§1.7).

### §1.1 Statement of the constraints

A uniform fleet of passenger cars provides railroad service to a set of cities. Service is offered by means of a set of scheduled "trains", each comprising one or more cars and running between a given pair of cities. At any given time, each car in the fleet is either part of some currently running train, or is sitting in storage at one of the cities.

Two requirements constrain the size and depolyment of the fleet: a fixed schedule, and known demands for scheduled trains.

Fixed schedule. The schedule lists all trains that depart in a chosen schedule-period (a day, for example). During the schedule-period, every scheduled train must be run, carrying one or more cars.

It is assumed that each schedule-period is followed immediately by another identical schedule-period. The same service is to be provided in every schedule-period: that is, the same schedule must be run, with the same allocation of cars to cities and trains.

Figure 1-1. Schedule and demands for a simple 2-city instance of the car-allocation problem. The schedule-period is a day. Note that the last train from B leaves at 23:00 each day and arrives at 2:00 the next day.

CITY A to CITY B

<u>Leave A</u>	<u>Arrive B</u>	<u>Demand</u>	<u>Cars required (75 pass./car)</u>
10:00	13:00	398	6
12:00	15:00	177	3
16:00	19:00	259	4
18:00	21:00	557	7
21:00	24:00	121	2

CITY B to CITY A

<u>Leave B</u>	<u>Arrive A</u>	<u>Demand</u>	<u>Cars required (75 pass./car)</u>
9:00	12:00	209	3
11:00	14:00	280	4
15:00	18:00	373	5
19:00	22:00	421	6
23:00	2:00	90	2

Each entry in the schedule specifies a city of departure and a city of arrival, and corresponding departure and arrival times. In general, a train may arrive during the schedule-period (e.g., day) of departure, or during any subsequent period. For simplicity, however, we assume here that every train arrives either in the same period, or at an earlier time in the next period. (If the schedule-period is a day, this just says that a train arrives either the same day that it leaves, or the next day; and that every trip lasts less than 24 hours.)

A car that arrives at city  $c$  at time  $t$  is free to leave  $c$  in any scheduled train that departs at  $t$  or later. Stopover delays at the arrival city -- to discharge and board passengers, for example -- are considered part of a train's trip and are reflected by adjusting the arrival time in the schedule accordingly.

Demands. For each scheduled train there is a known demand which must be met; hence there is a minimum number of cars required in each train. A train may be larger than its minimum size, however, if circumstances require that extra (deadhead) cars be shifted from one city to another.

Figure 1-1 shows a schedule and demands for a simple 2-city instance of this problem. Total demand from A to B requires 22 cars, while only 20 cars are required from B to A; consequently, in any feasible solution at least 2 extra cars will have to be deadheaded from B to A so that the stock of cars at A does not run out.

## §1.2 Formulation as a transshipment network

The train schedule is conveniently represented as a directed network whose unit of flow is one car. Nodes of the network correspond to the potential arrival or departure times at each city. Arcs represent the movement or storage of cars over time.

More specifically, partition the schedule-period into  $\tau$  uniform intervals beginning at times  $0, 1, \dots, \tau-1$ . (If the schedule-period is a day, time  $t$  could be the beginning of the  $t$ th minute of the day.) Describe

each train in the schedule by a departure city  $c$ , a departure time  $t \in \{0, \dots, \tau-1\}$ , an arrival city  $c'$ , and an arrival time  $t' \in \{0, \dots, \tau-1\}$ . Clearly the schedule may be made as precise as desired by choosing  $\tau$  sufficiently large.

Define one node in the network for each time in each city. If there are 4 cities and 1440 partitioning times, for example, the network has  $4 \cdot 1440$  nodes.

Connect the nodes by arcs of two types, representing cars in storage and cars in trains, respectively:

Storage arcs. For each city, run an arc from the node for each time  $t$  to the node for the next time,  $(t+1) \bmod \tau$ . The flow along such an arc represents cars held in storage at the city during the interval that begins at time  $t$ . (The last time,  $\tau-1$ , is connected to the first time, 0, since the last interval of the present schedule-period is followed immediately by the first interval of the next period.)

Train arcs. For each scheduled train, run an arc from the node representing the city and time of departure to the node for the city and time of arrival. Flow along this arc represents cars moving from one city to another in the scheduled train.

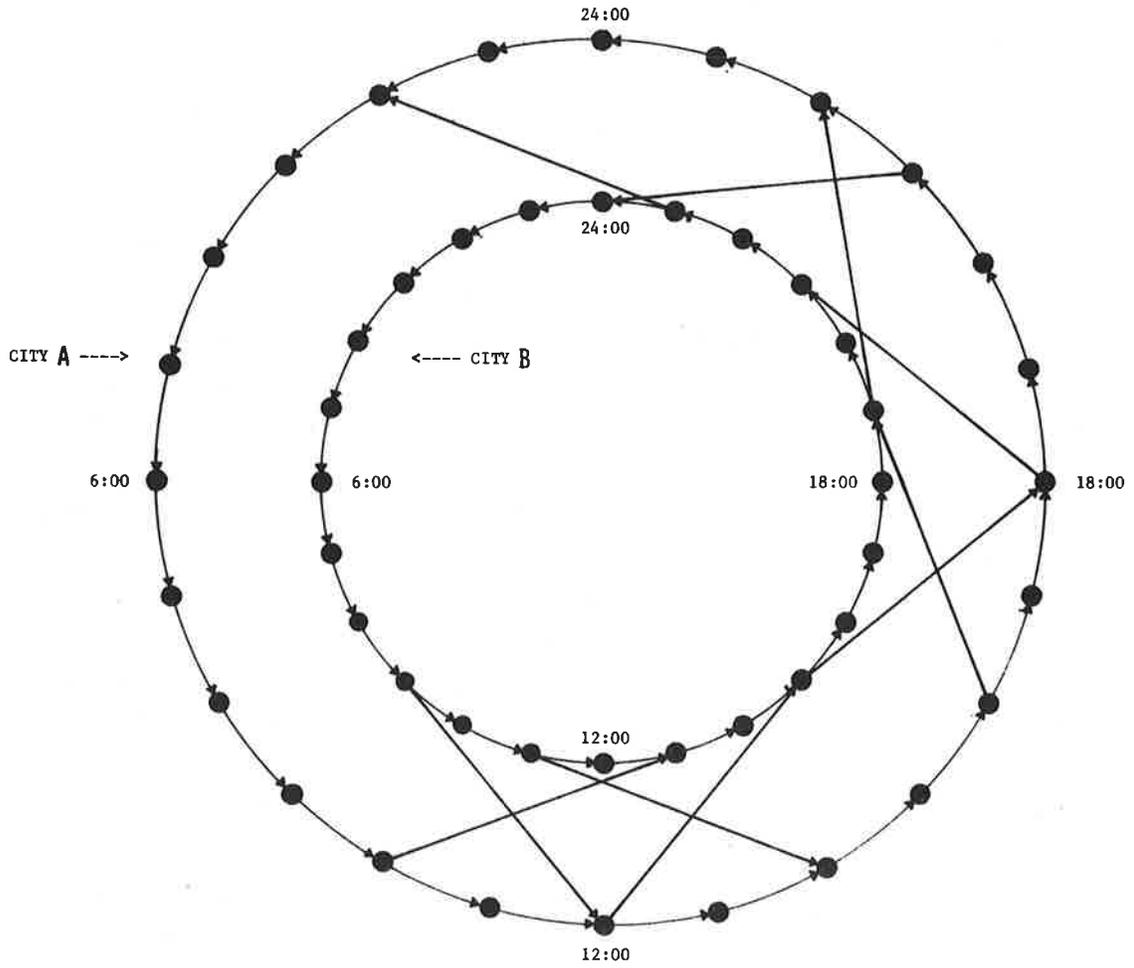
Flow around the network is constrained by the nature of the problem, in the following ways:

Conservation of flow. Since the fleet size is fixed, the number of cars in storage during interval  $t$  at a given city must equal the number in storage in the interval immediately before, plus the number that arrived at time  $t$ , less the number that departed at  $t$ . Equivalently, the net flow at every node must be zero: the network is built entirely of transshipment nodes.

Nonnegativity. All flows must be nonnegative. This amounts to requiring that trains cannot move backwards in time.

Integrality. Since cars are indivisible units, all flows must be integral.

Figure 1-2. A network equivalent of the sample problem. The day is divided into  $\tau = 24$  intervals, so that there is a node at each city at each hour.



Satisfaction of demand. The flow on each train arc must be greater than or equal to the number of cars needed to meet demand for the train. Demand thus places a lower limit on each arc. These lower limits are what force a positive flow around the network; they play the role of sources and sinks in more conventional transshipment-network formulations. (Indeed, an equivalent transshipment network without positive lower limits is easily constructed. One adds an appropriate sink for each departure at a node, and a source for each arrival.)

The network equivalent of Figure 1-1's example is shown in Figure 1-2.

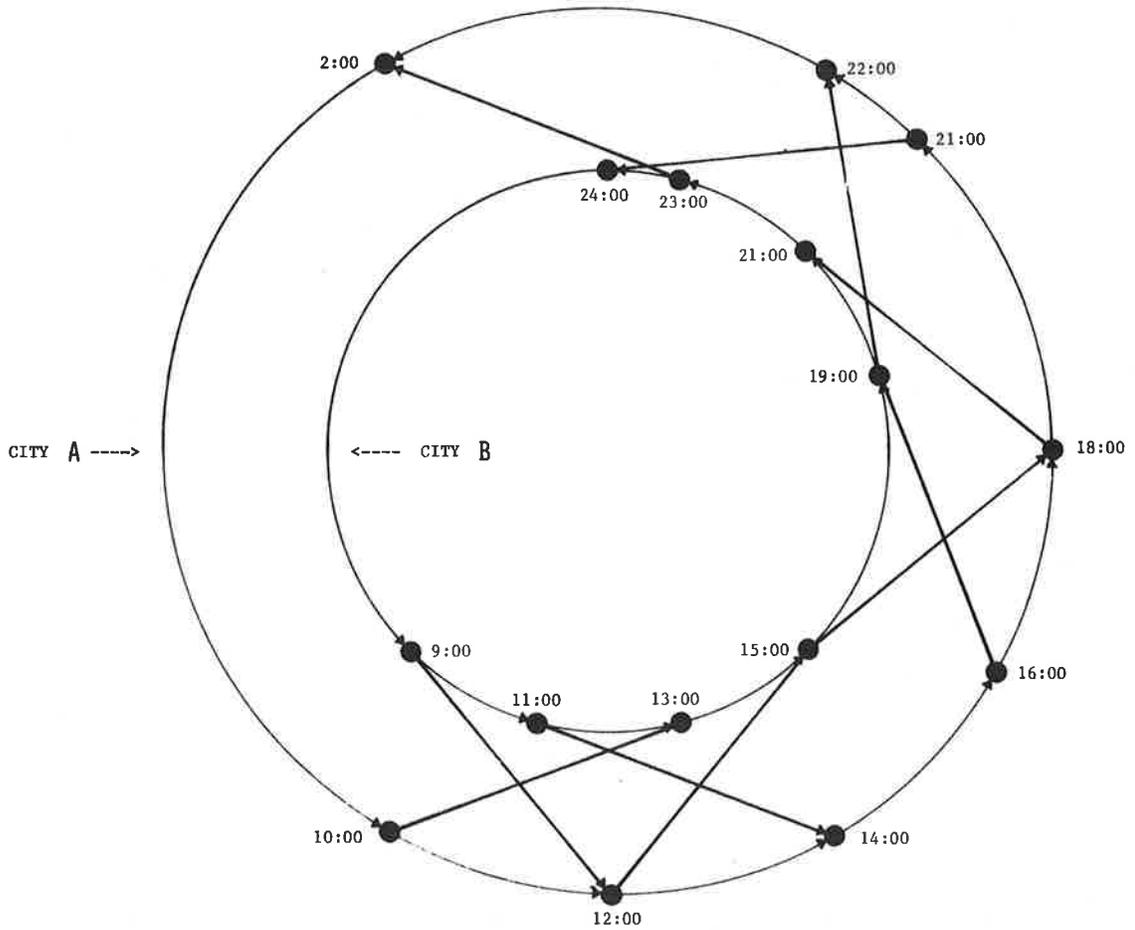
### §1.3 Reducing the network

If no trains arrive at or depart city  $c$  at time  $t$ , the node for  $c$  at  $t$  is connected to the rest of the network by only two storage arcs: an incoming arc from the previous time, and an outgoing arc to the following time. The flows on these two arcs must be the same in order to satisfy the conservation constraint. Consequently, one may remove the node and replace the two arcs with one. Other flows in the network are as before, and remain feasible if they were previously so; hence this transformation leaves the set of feasible solutions essentially unchanged.

When all such "inactive" nodes are removed, there remains a network of minimum size for the problem. Figure 1-3 shows a reduced network of this sort, for the problem of Figures 1-1 and 1-2. When the number of intervals  $\tau$  is quite large (the number of minutes in a day, for instance), reducing the network to active nodes is imperative if the network is to be kept to a manageable size. All cases run in the studies discussed later in this report employed reduced networks.

It is possible to formulate the problem directly in terms of finite sets of active nodes, one set for each city chosen from an interval of time  $[0, \tau)$ . To promote simplicity of notation, however, the results of the following sections are expressed in terms of unreduced networks.

Figure 1-3. The reduced equivalent of the network.



#### §1.4 Formulation as linear constraints

Any network of the sort we have outlined may be described by an equivalent linear-programming model. To each arc of the network there corresponds a structural variable, whose activity equals the arc's flow. Conservation constraints on flows become linear equalities in the variables, while common LP techniques implicitly guarantee nonnegativity, integrality, and satisfaction of demand at every feasible basic solution.

To express the LP formally, define the following sets:

$C$  the set of cities  
 $T = \{0, \dots, \tau-1\}$  the set of intervals into which the schedule-period is divided  
 $S \subset \{(c,t,c',t') \in C \times T \times C \times T \mid c \neq c'\}$   
the schedule: each element represents a train that leaves city  $c$  at time  $t$  and arrives at city  $c'$  at  $t'$

Represent the demands by

$d_{cc'}[t,t'] > 0$  the smallest (integral) number of cars required to meet demand for train  $(c,t,c',t') \in S$

Express the nodes of the network as:

$A_c[t]$  for all  $c \in C, t \in T$

The directed arcs representing storage of unused cars are then

$u_c[t]: A_c[t] \rightarrow A_c[(t+1) \bmod \tau]$  for all  $c \in C, t \in T$

The arcs representing movement of cars in trains are

$x_{cc'}[t,t']: A_c[t] \rightarrow A_{c'}[t']$  for all  $(c,t,c',t') \in S$

Define an LP structural variable corresponding to each arc, and representing the flow over the arc:

$u_c[t]$  flow over  $u_c[t]$ , for all  $c \in C, t \in T$

$x_{cc'}[t,t']$  flow over  $x_{cc'}[t,t']$ , for all  $(c,t,c',t') \in S$

The constraints on network flow are expressed as follows:

Conservation of flow:

$$\begin{aligned}
 u_c[(t-1) \bmod \tau] + \sum_{(c_1, t_1, c, t) \in S} x_{c_1 c}^{[t_1, t]} \\
 = u_c[t] + \sum_{(c, t, c_2, t_2) \in S} x_{c c_2}^{[t, t_2]} \quad \text{for all } c \in C, t \in T
 \end{aligned}$$

Satisfaction of demand:

$$x_{cc'}^{[t, t']} \geq d_{cc'}^{[t, t']} \quad \text{for all } (c, t, c', t') \in S$$

Nonnegativity:

$$u_c[t] \geq 0 \quad \text{for all } c \in C, t \in T$$

Integrality:

$$\begin{aligned}
 u_c[t] \text{ integral} & \quad \text{for all } c \in C, t \in T \\
 x_{cc'}^{[t, t']} \text{ integral} & \quad \text{for all } (c, t, c', t') \in S
 \end{aligned}$$

Nonnegativity of the  $x$  variables is insured by satisfaction of demand.

Given that all  $d_{cc'}^{[t, t']}$  are integral, a fundamental property of transportation problems guarantees that every basic solution to the above LP is an integral solution. Consequently, a feasible solution to the above problem -- and hence a feasible allocation of cars to trains -- may be determined directly by application of the (phase 1) simplex method. Given any linear objective function, the simplex method will also find an optimal feasible allocation.

Both satisfaction of demand and nonnegativity express simple lower bounds on the variables. Constraints of this sort are easily handled implicitly by the simplex method. Hence only the conservation-of-flow equations need appear explicitly as rows in the LP.

### §1.5 Corridor service and turnaround delays

A "corridor" is a set of cities related by a directional ordering that is complete, transitive, and irreflexive. In other words, the cities of a corridor may be indexed  $a_1, a_2, \dots, a_n$ , such that  $a_i$  is in the given direction from  $a_j$  if and only if  $i > j$ . The Northeast Corridor is a corridor in this sense, ordered by the relation "north of".

Every train in a corridor must run in the ordering direction, or in the opposite direction. For convenience, these directions are here called north and south; they could just as well be east and west, or clockwise and counterclockwise. Trains are thus labeled northbound or southbound, accordingly.

In our initial formulation, stopover delay at the arrival city is implicit in the schedule and, therefore, it is the same for every car in a train. Within a corridor, however, it is reasonable to specify that the stopover delay for a car that changes direction is some number of intervals greater than the delay for a car that continues in the same direction along the corridor. Thus cars in a train from, say, Philadelphia to New York may continue to move north, after a minimal stop, in a train from New York to Boston; but cars in the Philadelphia-New York train that are to be taken off and sent back to Philadelphia are delayed in New York for a somewhat longer time. A similar "turnaround delay" is encountered in resuming service after one end of the corridor (say, Boston) is reached.

Turnaround delays cannot be modeled by simply adjusting the schedule because, in general, some cars in a train may continue in the same direction, while others are detached and turned around. A simple and feasible approach, however, is to duplicate the original network, creating two separate but similar parts: one for northbound trains, and one for southbound trains. Arcs connecting the two parts are added to represent cars being turned around.

Specifically, partition the schedule into two sets  $S^N$  and  $S^S$  of northbound and southbound trains, respectively. For the northbound trains, construct a full network as before:

$$A_c^N[t] \quad \text{for all } c \in C, t \in T$$

(nodes representing potential arrival and departure times of northbound trains)

$$u_c^N[t]: A_c^N[t] \rightarrow A_c^N[(t+1) \bmod \tau] \quad \text{for all } c \in C, t \in T$$

(arcs representing unused northbound cars  
in storage at each city and time)

$$x_{ce}, [t, t']: A_c^N[t] \rightarrow A_{c'}^N[t'] \quad \text{for all } (c, t, c', t') \in S^N$$

(arcs representing cars moving in northbound trains)

In the same way, define a separate network for southbound trains:

$$A_c^S[t] \quad \text{for all } c \in C, t \in T$$

$$u_c^S[t]: A_c^S[t] \rightarrow A_c^S[(t+1) \bmod \tau] \quad \text{for all } c \in C, t \in T$$

$$x_{ce}, [t, t']: A_c^S[t] \rightarrow A_{c'}^S[t'] \quad \text{for all } (c, t, c', t') \in S^S$$

Represent the number of intervals required to change a car's direction by  $\delta$ . Connect the northbound and southbound networks by two sets of arcs that represent unused cars in storage that are being turned around:

$$u_c^{NS}[t]: A_c^N[t] \rightarrow A_c^S[(t+\delta) \bmod \tau] \quad \text{for all } c \in C, t \in T$$

(arcs representing formerly northbound cars, in  
storage at time  $t$ , that will be switched to run  
south  $\delta$  intervals later)

$$u_c^{SN}[t]: A_c^S[t] \rightarrow A_c^N[(t+\delta) \bmod \tau] \quad \text{for all } c \in C, t \in T$$

(arcs representing formerly southbound cars, in  
storage at time  $t$ , that will be switched to run  
north  $\delta$  intervals later)

The construction of these connecting arcs guarantees that northbound cars reaching city  $c$  at time  $t$  must wait at least  $\delta$  intervals before they can be incorporated in a southbound train.

The constraints on this expanded network are analogous in every respect to those on the original one: flow must be conserved at all nodes, all flows must be nonnegative and integral, and demand must be satisfied along the  $X^N$  and  $X^S$  arcs. As before, the network has a transshipment structure, and can be modeled by a linear program all of whose basic solutions are integral.

For practical purposes, one can apply the methods of this section to the reduced network of §1.3, to produce separate reduced northbound and southbound networks having a reduced set of connecting arcs. (The full network cannot, in general, be reduced directly without the loss of some feasible solutions. However, under reasonable assumptions about the objective function, one may be able to show that the eliminated feasible solutions perform unneeded or uneconomical changes of direction, and cannot be optimal. Some direct reduction of the network would then be valid.)

The methods of this section are not fundamentally limited to the case of a single, fixed turnaround delay. One could easily incorporate a set of delays that vary with time, city, or direction, by making appropriate changes to the definitions of the  $u^{NS}$  and  $u^{SN}$  arcs. Extensions of these methods might also be applied to sets of cities that are not corridors.

#### §1.6 Upper limits on train sizes

The model developed so far insures only that each train is allocated enough cars. One may also wish to specify that it is not allocated too many. For example, the number of cars in a train could be limited to twice the number needed to meet demand, to keep load factors at reasonable levels. Stations' platform lengths might also dictate some absolute bound on train sizes.

Upper limits are easily incorporated in the linear programs of §1.4 or §1.5. Define

$$h_{cc'}[t,t'] \geq d_{cc'}[t,t']$$

as the maximum feasible size of the train  $(c,t,c',t') \in S$ . Then the constraints on the  $x$  variables in the linear program are augmented to

$$d_{cc'}[t,t'] \leq x_{cc'}[t,t'] \leq h_{cc'}[t,t']$$

for all  $(c,t,c',t') \in S$ .

Upper limits of this sort do not destroy the model's transshipment structure. Hence all basic solutions are still integral, and the simplex

method may be applied as before. Moreover, the augmented constraints on the  $x$  variables are still simple bounds that can be handled implicitly by the simplex method; the number of explicit rows in the LP is unchanged.

#### §1.7 Modeling locomotive requirements

In general, the number of locomotives required to haul a scheduled train depends on the number of cars assigned to the train. Since the number of cars may vary between feasible solutions, so may the number of locomotives.

By judicious choice of upper limits  $h_{cc'}[t,t']$  (§1.6), however, one may be able to restrict the size of each train  $(c,t,c',t') \in S$  so that its requirement for locomotives,  $e_{cc'}[t,t']$ , is fixed. Then the flow of locomotives may be modeled in exactly the same way as the flow of cars. One simply replaces car demands  $d_{cc'}[t,t']$  in §§1.1-1.5 by the locomotive demands  $e_{cc'}[t,t']$ . Upper limits on the number of locomotives pulling each train may also be imposed, in the manner of §1.6.

Any of the optimization techniques described in §2 may be applied to the locomotive-demand case. Many of the results expressed in terms of cars are also meaningful in terms of locomotives.

Application of these ideas to locomotive requirements in the Northeast Corridor is described in §4.7.



A feasible set of car allocations for the problem formulated in §1 -- if such a set exists -- may be determined by application of the simplex algorithm, phase 1. Given that a feasible allocation exists, the next step is to seek an allocation that optimizes some functional in the  $x$  and  $u$  variables. For purposes of this report we are concerned with functionals of one particularly useful and tractable sort: linear objective functions related to costs.

Minimizing cost is a natural objective for any planning model. Since our network model, in particular, fixes the level of service and requires that all demands be met, cost is the principal criterion of difference between feasible allocations. In addition, certain classes of minimum-cost solutions may be characterized in particularly revealing ways.

Linear functionals have a purely practical justification: they may be minimized by straightforward application of the simplex method. Fortunately, several reasonable measures of cost are proportional to linear functionals, as shown in §2.1.

Approaches to minimizing more than one linear cost objective are discussed in §2.2. The case of two objectives is developed in §§2.3-2.4, and the results are applied in §2.5 to two objectives of particular interest.

For convenience of exposition, the schedule-period is hereafter taken to be a day. A set of solution activities of the  $x$  and  $u$  variables is written  $(x,u)$ , and the value of a functional  $Z$  at the solution is  $Z(x,u)$ .

### §2.1 Linear functionals representing costs

There is more than one sort of cost associated with railroad service, and consequently one may devise a number of linear forms that are proportional

to cost of some sort. Three functionals of particular interest - associated with capital, operating, and switching costs, respectively -- are formulated as follows:

Capital cost. The daily cost of amortizing the passenger-car fleet, here referred to as the "capital cost", may be considered proportional to the number of cars in the fleet. Hence minimizing fleet size serves to minimize capital cost.

The number of cars in the system is easily represented by a linear form. Pick any time  $t^*$ ,  $0 \leq t^* \leq \tau-1$ , and sum (a) the number of cars in storage at each city in interval  $t^*$ , and (b) the number of cars in each train that is in transit during interval  $t^*$ . This sum is the total number of cars in the system at  $t^*$ . For a feasible solution, this sum must be the same at any  $t^*$  since cars may not enter or leave the system. For convenience, take  $t^* = \tau-1$ ; then the capital-cost objective is a linear combination

$$Z_{\text{CAR}} = \sum_{c \in C} u_c [\tau-1] + \sum_{\substack{(c,t,c',t') \in S \\ t' < t}} x_{cc'} [t,t']$$

The first sum covers all cars in storage during interval  $\tau-1$ . The latter counts cars in only those trains which depart during one day and arrive the next: these are exactly the trains that are in transit during the last interval,  $\tau-1$ , of the day.

Operating cost. Costs proportional to the number of car-miles run in a day, here called "operating costs", are another logical candidate for minimization. Letting the distance from  $c$  to  $c'$  be  $m_{cc'}$ , total car-miles per day is equal to the linear form

$$Z_{\text{MILE}} = \sum_{(c,t,c',t') \in S} m_{cc'} x_{cc'} [t,t']$$

Note that at any feasible solution  $Z_{\text{MILE}}$  is also a sum of integral multiples of the distances  $m_{cc'}$ . Moreover, when the cities form a corridor (§1.5),  $Z_{\text{MILE}}$  is a sum of integral multiples of the round-trip distances:

$$m_{c_i c_j} + m_{c_j c_i}, \quad i < j$$

since conservation of the flow of cars requires that the number of cars run north from  $c_i$  to  $c_j$  during a day is the same as the number run south from  $c_j$  to  $c_i$ .

$Z_{\text{MILE}}$  is also closely related to load factor. Given fixed demands, it is reasonable to try to maximize system load factor in order to minimize the cost of providing service. By definition, system load factor is

$$\begin{aligned} Z_{\text{LF}} &= \frac{\text{passenger-miles / day}}{\text{seat-miles / day}} \\ &= \frac{(\text{passenger-miles / day}) / (\text{seats / car})}{\text{car-miles / day}} \end{aligned}$$

Since both passenger-miles/day and seats/car are fixed by the problem,  $Z_{\text{LF}}$  is inversely proportional to car-miles/day =  $Z_{\text{MILE}}$ . Hence minimizing operating cost is equivalent to maximizing the system load factor.

Switching cost. For the corridor model of §1.5, one may postulate an extra fixed "switching" cost incurred each time a car's direction is reversed. The number of car-reversals in a day is counted by the following linear form:

$$Z_{\text{TURN}} = \sum_{c \in C, t \in T} u_c^{\text{NS}}[t] + \sum_{c \in C, t \in T} u_c^{\text{SN}}[t]$$

The first term sums all northbound cars turned south, and the second all southbound cars turned north.

## §2.2 Combining measures of cost

It was shown in §2.1 that there are several reasonable "costs" that are proportional to linear functionals in the  $u$  and  $x$  variables. As a consequence, no solution that merely minimizes one of these functionals is entirely satisfactory. For example, an allocation that minimizes the number of cars (capital cost) may nonetheless employ them inefficiently, running them more than the minimum car-miles/day (operating cost).

Some means is needed, therefore, of optimizing with respect to more than one cost objective. Two methods suggest themselves: combining objectives so that they are minimized simultaneously, and ordering objectives so that they may be minimized successively.

Combining objectives. Any  $n$  objective functions  $Z_1, Z_2, \dots, Z_n$  can be combined by choosing factors  $p_1, p_2, \dots, p_n \geq 0$ , and minimizing the linear combination

$$Z = p_1 Z_1 + p_2 Z_2 + \dots + p_n Z_n$$

Minimizing  $Z$  tends to minimize each of the  $Z_i$ . The value of  $Z_i$  at min  $Z$  is, however, generally greater than  $\min Z_i$ ; the extent of the discrepancy depends on the size of  $p_i$  with respect to the other factors.

$Z$  has a natural interpretation when there is some cost proportional to each  $Z_i$ . Let  $p_i$  be the constant of proportionality, so that  $p_i Z_i$  is the cost (in dollars, say) corresponding to any given level of  $Z_i$ . (If  $Z_1$  is car-miles/day, for example,  $p_1$  could be operating expense in dollars/car-mile.)  $Z$  is thus a "total variable cost" for the system, and minimizing  $Z$  can be seen as minimizing total cost.

The difficulty with this approach lies in determining true values for the constants  $p_i$ . Even small changes to the  $p_i$  can produce significant differences in the solution to min  $Z$ ; yet, especially when a hypothetical system is being modeled, costs are often poorly known and the  $p_i$  can be determined only to within a wide tolerance. Hence it is necessary to treat the  $p_i$  as somewhat variable, and to find solutions for ranges of their values. (An efficient and exhaustive way of doing this when total cost is the sum of two costs is described in the following section.)

Ordering objectives. Another approach is to rank the objectives, minimizing  $Z_i$  subject to  $Z_1, \dots, Z_{i-1}$  being fixed at their previously attained values. One first computes  $\min Z_1$ , the absolute minimum value of  $Z_1$ ; then  $\min Z_2 | Z_1$ , the minimum value of  $Z_2$  given  $Z_1 = \min Z_1$ ; then  $\min Z_3 | Z_2 | Z_1$ , the minimum value of  $Z_3$  given  $Z_2 = \min Z_2 | Z_1$  and  $Z_1 = \min Z_1$ ; and so forth. In general,  $\min Z_i | Z_{i-1} | \dots | Z_1$  is greater than the absolute  $\min Z_i$ , and the discrepancy tends to become greater as  $i$  does.

A solution to  $\min Z_2 | Z_1$  is found, in effect, by adding a new equality constraint ( $Z_1 = \min Z_1$ ). The original problem's pure transshipment structure is thus violated. Nevertheless, an optimal integral solution is guaranteed by the following Proposition.

Proposition 1. For any linear forms  $Z_1(x,u)$  and  $Z_2(x,u)$  there is an integral basic solution to  $\min Z_2|Z_1$ .

Proof: The transshipment structure of the original problem guarantees that there is a finite set of integral basic solutions to  $\min Z_1$ :

$$(x_1^*, u_1^*), \dots, (x_n^*, u_n^*)$$

A standard result of linear programming asserts that *any* solution  $(x^*, u^*)$  at which  $Z_1 = \min Z_1$  is a convex combination of these basic solutions:

$$(x^*, u^*) = \sum \lambda_i (x_i^*, u_i^*), \quad \sum \lambda_i = 1$$

Hence  $\min Z_2|Z_1$  is necessarily the minimum of  $Z_2$  over all solutions  $(x^*, u^*)$  having this form. However, the linearity of  $Z_2$  insures that for any  $(x^*, u^*)$

$$Z_2(x^*, u^*) = \sum \lambda_i Z_2(x_i^*, u_i^*) \geq Z_2(x_k^*, u_k^*)$$

for some  $k$ . Thus  $\min Z_2|Z_1$  must occur at some basic solution to the original problem, and this is the integral solution desired.

This proof is easily extended to show that  $\min Z_n|Z_{n-1}|\dots|Z_1$  has an integral basic solution for any  $n$ .

Sequential optimization has the advantage of requiring only a preferential ordering of costs, rather than a full determination of their relative sizes. It is disadvantageous primarily in being less general than the "total cost" approach above. (The two approaches are closely related, however, as shown in §2.4.)

### §2.3 The case of two objective functions

When attention is restricted to two cost objectives, the set of all possible optimal allocations can be described in a simple way. Moreover, the representative optima are easily found by use of an algorithm for parametric programming on the objective.

Denote the two objectives by  $Y$  and  $Z$ , and their respective expenses per unit by  $p_Y$  and  $p_Z$ . A total cost determined by  $Y$  and  $Z$  is thus  $p_Y Y + p_Z Z$ . The minimum total cost is:

$$\begin{aligned} \min [p_Y Y + p_Z Z] &= p_Y \min [Y + (p_Z/p_Y) Z] \\ &= p_Y \min [Y + \rho Z] \end{aligned}$$

where  $\rho = p_Z/p_Y$  is the ratio of expenses per unit. Hence the minimum total cost is determined entirely by the choice of  $\rho$ .

The set of all solutions that can minimize total cost, given some choice of  $\rho$ , is characterized in the following Propositions:

Proposition 2. If  $(x^*, u^*)$  minimizes  $Y + \rho Z$ ,  $\rho \geq 0$ , then  $R = \{k \geq 0 \mid (x^*, u^*) \text{ minimizes } Y + kZ\}$  satisfies one of

- (i)  $R = \{\rho\}$
- (ii)  $R = [r, s]$  for some  $r \leq \rho \leq s$ ,  $0 \leq r < s \leq \infty$

Proof: Suppose that  $(x^*, u^*)$  minimizes both  $Y + \rho_1 Z$  and  $Y + \rho_2 Z$ ,  $\rho_1 < \rho_2$ . Then, for any  $\rho$  satisfying  $\rho_1 \leq \rho \leq \rho_2$ ,  $(x^*, u^*)$  also minimizes the convex combination

$$\frac{\rho_2 - \rho}{\rho_2 - \rho_1} (Y + \rho_1 Z) + \frac{\rho - \rho_1}{\rho_2 - \rho_1} (Y + \rho_2 Z) = Y + \rho Z$$

Hence if  $R \neq \{\rho\}$ , it must be an interval from some  $r \leq \rho_1$  to some  $s \geq \rho_2$ ,  $r \neq s$ . Continuity of the problem guarantees that the interval is closed.

Proposition 3. Let

$$R_1 = \{k > 0 \mid (x_1^*, u_1^*) \text{ minimizes } Y + kZ\} \neq \emptyset$$

$$R_2 = \{k > 0 \mid (x_2^*, u_2^*) \text{ minimizes } Y + kZ\} \neq \emptyset$$

Then exactly one of the following is true:

- (i)  $R_1 \cap R_2 = \emptyset$
- (ii)  $R_1 \cap R_2 = \{\rho\}$ ,  $\rho \geq 0$
- (iii)  $R_1 = R_2 = [r, s]$ ,  $0 \leq r < s \leq \infty$

Proof: Suppose neither (i) nor (ii) holds.  $R_1 \cap R_2$  thus contains more than one point, and consequently both  $R_1$  and  $R_2$  contain more than one point. Hence by Proposition 2

$$R_1 = [r_1, s_1] \quad 0 \leq r_1 < s_1 \leq \infty$$

$$R_2 = [r_2, s_2] \quad 0 \leq r_2 < s_2 \leq \infty$$

Let  $\rho_1$  and  $\rho_2$  be distinct points in  $R_1 \cap R_2$ . Then both  $(x_1^*, u_1^*)$  and  $(x_2^*, u_2^*)$  minimize both  $Y + \rho_1 Z$  and  $Y + \rho_2 Z$ ; that is:

$$Y(x_1^*, u_1^*) + \rho_1 Z(x_1^*, u_1^*) = Y(x_2^*, u_2^*) + \rho_1 Z(x_2^*, u_2^*)$$

$$Y(x_1^*, u_1^*) + \rho_2 Z(x_1^*, u_1^*) = Y(x_2^*, u_2^*) + \rho_2 Z(x_2^*, u_2^*)$$

Subtracting the second equation from the first yields:

$$(\rho_1 - \rho_2) Z(x_1^*, u_1^*) = (\rho_1 - \rho_2) Z(x_2^*, u_2^*)$$

Since  $\rho_1 - \rho_2 \neq 0$ ,  $Z(x_1^*, u_1^*) = Z(x_2^*, u_2^*)$ , and consequently also  $Y(x_1^*, u_1^*) = Y(x_2^*, u_2^*)$ . Hence any objective of the form  $Y + kZ$  minimized by  $(x_1^*, u_1^*)$  is also minimized by  $(x_2^*, u_2^*)$ , and vice versa; that is  $R_1 = R_2 = [r, s]$  as required by (iii).

Proposition 4. Let  $Y$  and  $Z$  be objectives for which  $\min Y$  and  $\min Z$  are finite. For any  $(x^*, u^*)$ , define:

$$R_{x^*, u^*} = \{k \geq 0 \mid (x^*, u^*) \text{ minimizes } Y + kZ\}$$

Then:

(a) There is a unique sequence

$$0 = \rho_0, \rho_1, \dots, \rho_{n-1}, \rho_n = \infty, \quad n > 1; \rho_{i-1} < \rho_i, i = 1, \dots, n$$

and there is a corresponding set of distinct basic solutions

$$(x_i^*, u_i^*) \quad i = 1, \dots, n$$

such that

$$R_{x_i^*, u_i^*} = [\rho_{i-1}, \rho_i] \quad i = 1, \dots, n$$

(b) For any solution  $(x^*, u^*)$ , exactly one of the following holds:

- (i)  $R_{x^*, u^*} = \phi$
- (ii)  $R_{x^*, u^*} = \{\rho_i\}$ , for some  $i \in \{0, \dots, n-1\}$
- (iii)  $R_{x^*, u^*} = [\rho_{i-1}, \rho_i]$ , for some  $i \in \{1, \dots, n\}$

(c) For every  $i = 1, \dots, n-1$ ,

$$Y(x_i^*, u_i^*) < Y(x_{i+1}^*, u_{i+1}^*)$$

$$Z(x_i^*, u_i^*) > Z(x_{i+1}^*, u_{i+1}^*)$$

Proof: (a) Consider the set of all basic solutions to  $Y + kZ$ , for any  $k \geq 0$ :

$$B = \{\text{basic } (x^*, u^*) \mid (x^*, u^*) \text{ minimizes } Y + kZ \text{ for at least one } k \geq 0\}$$

$B$  is finite, since there is a finite number of bases. Moreover, since  $\min Y$  and  $\min Z$  have finite solutions,  $\min Y + kZ$  has a finite solution for any  $k \geq 0$ , and hence  $\min Y + kZ$  has a basic solution for any  $k \geq 0$ . Equivalently

$$\bigcup_{(x^*, u^*) \in B} R_{x^*, u^*} = [0, \infty]$$

By Propositions 2 and 3, this is a union of single points and closed intervals having disjoint interiors. Hence, since  $B$  is finite, there is a minimal subset  $\{(x_i^*, u_i^*), i = 1, \dots, n\} = B^* \subset B$  such that

$$\bigcup_{(x^*, u^*) \in B^*} R_{x^*, u^*} = [0, \infty]$$

is a union of distinct closed intervals having disjoint interiors.  $B^*$  is thus the desired set of distinct basic solutions.

The uniqueness of the sequence  $\rho_1, \dots, \rho_{n-1}$  is guaranteed by Proposition 3.

(b) Consider any solution  $(x^*, u^*)$ . By Proposition 2, there are three possibilities:

$R_{x^*, u^*} = \emptyset$ . This satisfies case (i).

$R_{x^*, u^*} = [r, s]$ ,  $0 \leq r < s \leq \infty$ . Proposition 3 requires that  $[r, s] = [\rho_{i-1}, \rho_i]$  for some choice of  $i = 1, \dots, n$ , satisfying case (iii).

$R_{x^*, u^*} = \{\rho\}$ . For some  $i \leq n$ ,  $\rho_{i-1} \leq \rho \leq \rho_i$ . Assume that the inequalities are strict; then  $(x_i^*, u_i^*)$  minimizes both  $Y + \rho_{i-1}Z$  and  $Y + \rho_i Z$ , while  $(x^*, u^*)$  minimizes neither, implying that

$$Y(x_i^*, u_i^*) + \rho_{i-1} Z(x_i^*, u_i^*) < Y(x^*, u^*) + \rho_{i-1} Z(x^*, u^*)$$

$$Y(x_i^*, u_i^*) + \rho_i Z(x_i^*, u_i^*) < Y(x^*, u^*) + \rho_i Z(x^*, u^*)$$

Multiplying the first inequality by  $(\rho_i - \rho) / (\rho_i - \rho_{i-1})$ , and the second by  $(\rho - \rho_{i-1}) / (\rho_i - \rho_{i-1})$ , and adding, yields:

$$Y(x_i^*, u_i^*) + \rho Z(x_i^*, u_i^*) < Y(x^*, u^*) + \rho Z(x^*, u^*)$$

contradicting  $R_{x^*, u^*} = \{\rho\}$ . Hence  $\rho = \rho_{i-1}$  or  $\rho = \rho_i$ , demonstrating (ii).

(c) Both  $(x_i^*, u_i^*)$  and  $(x_{i+1}^*, u_{i+1}^*)$  minimize  $Y + \rho_i Z$ ; therefore

$$Y(x_i^*, u_i^*) + \rho_i Z(x_i^*, u_i^*) = Y(x_{i+1}^*, u_{i+1}^*) + \rho_i Z(x_{i+1}^*, u_{i+1}^*)$$

But for any  $\rho$  such that  $\rho_i < \rho \leq \rho_{i+1}$ ,  $(x_{i+1}^*, u_{i+1}^*)$  minimizes  $Y + \rho Z$  and  $(x_i^*, u_i^*)$  does not, implying

$$Y(x_i^*, u_i^*) + \rho Z(x_i^*, u_i^*) > Y(x_{i+1}^*, u_{i+1}^*) + \rho Z(x_{i+1}^*, u_{i+1}^*)$$

Subtracting the equation from the inequality and dividing by  $\rho - \rho_i > 0$  yields

$$Z(x_i^*, u_i^*) > Z(x_{i+1}^*, u_{i+1}^*)$$

and  $Y(x_i^*, u_i^*) < Y(x_{i+1}^*, u_{i+1}^*)$  follows by substituting back into the equation.

What do the values  $\rho_i$  signify? They are the *critical ratios*  $p_Z/p_Y$  at which the allocation of trains must change to maintain optimal total cost. So long as  $p_Z/p_Y$  stays between some  $\rho_{i-1}$  and  $\rho_i$ , however, a single allocation  $(x_i^*, u_i^*)$  is guaranteed optimal.

Another way of looking at things is to note that, at critical point  $p_Z/p_Y = \rho_i$ ,

$$Y(x_i^*, u_i^*) + \rho_i Z(x_i^*, u_i^*) = Y(x_{i+1}^*, u_{i+1}^*) + \rho_i Z(x_{i+1}^*, u_{i+1}^*)$$

which may be rewritten

$$p_Y [Y(x_{i+1}^*, u_{i+1}^*) - Y(x_i^*, u_i^*)] = p_Z [Z(x_i^*, u_i^*) - Z(x_{i+1}^*, u_{i+1}^*)]$$

Proposition 4(c) says that moving from  $(x_i^*, u_i^*)$  to  $(x_{i+1}^*, u_{i+1}^*)$  involves a tradeoff:  $Z$  decreases while  $Y$  increases. At the critical point, the added cost from the increase in  $Y$  (left side of above equation) equals the cost saved by decreasing  $Z$  (right side). At  $\rho < \rho_i$ , the saved cost does not make up the added cost, and so  $(x_i^*, u_i^*)$  is preferable; at  $\rho > \rho_i$ , the saved cost more than makes up the added cost, and hence  $(x_{i+1}^*, u_{i+1}^*)$  is better.

The critical ratios  $\rho_i$  and solutions  $(x_i^*, u_i^*)$  are easily found by applying a standard parametric algorithm to the objective. In conventional terms,  $Y$  is the "base objective" and  $Z$  the "change objective". The algorithm starts with a solution for  $\min Y$ , and "parameter"  $\rho$  at 0. Successive pivots either leave  $\rho$  unchanged, or step it to a new critical value that is generally one of the critical ratios  $\rho_i$ ; the basis just before the step to  $\rho_i$  is  $(x_i^*, u_i^*)$ . The algorithm terminates when it finds a solution that is optimal for all parameter values greater than some critical value; this solution is  $(x_n^*, u_n^*)$ , and the critical value is  $\rho_{n-1}$ .

In some instances, the algorithm identifies a supposed critical ratio  $\rho_i$  such that

$$\begin{aligned} Y(x_i^*, u_i^*) &= Y(x_{i+1}^*, u_{i+1}^*) \\ Z(x_i^*, u_i^*) &= Z(x_{i+1}^*, u_{i+1}^*) \end{aligned}$$

This cannot be a critical ratio, however, since the above equalities violate Proposition 4(c). Indeed, these equalities imply that both  $(x_i^*, u_i^*)$  and  $(x_{i+1}^*, u_{i+1}^*)$  minimize  $Y + \rho Z$  for all  $\rho$  such that  $\rho_{i-1} \leq \rho \leq \rho_{i+1}$ , so that  $\rho_i$  is actually not critical at all. Spurious ratios of this sort are a side effect of degeneracy in the linear program.

#### §2.4 Conditional optima for the case of two objectives

The solutions  $(x_i^*, u_i^*)$  derived in Proposition 4 also have an interpretation in terms of  $\min Z|Y$ ,  $\min Y|Z$ , and other conditional optima. This is shown in the following Proposition:

Proposition 5. The solutions  $(x_i^*, u_i^*)$  defined in Proposition 4 have the following properties:

- (a)  $(x_1^*, u_1^*)$  minimizes  $Y$   
 $(x_1^*, u_1^*)$  minimizes  $Z|Y$
- (b)  $(x_n^*, u_n^*)$  minimizes  $Z$   
 $(x_n^*, u_n^*)$  minimizes  $Y|Z$
- (c)  $(x_i^*, u_i^*)$  minimizes  $Z|(Y + \rho Z)$ ,  $\rho_{i-1} \leq \rho < \rho_i$  for  $i = 1, \dots, n$   
 $(x_i^*, u_i^*)$  minimizes  $Y|(Y + \rho Z)$ ,  $\rho_{i-1} < \rho \leq \rho_i$  for  $i = 1, \dots, n$

Proof: (a) By Proposition 4,  $(x_1^*, u_1^*)$  minimizes  $Y + \rho Z$  when  $\rho = 0$ , hence it minimizes  $Y$ . Given any other  $(x^*, u^*)$  that minimizes  $Y$ , and any positive  $\rho < \rho_1$ ,

$$\begin{aligned} Y(x_1^*, u_1^*) + \rho Z(x_1^*, u_1^*) &\leq Y(x^*, u^*) + \rho Z(x^*, u^*) \\ Y(x_1^*, u_1^*) &= Y(x^*, u^*) \end{aligned}$$

(the former from the fact that  $(x_1^*, u_1^*)$  minimizes  $Y + \rho Z$ , the latter since both solutions minimize  $Y$ ). Subtracting and dividing by  $\rho > 0$  yields

$$Z(x_1^*, u_1^*) \leq Z(x^*, u^*)$$

This is true for any  $(x^*, u^*)$  minimizing  $Y$ ; hence  $(x_1^*, u_1^*)$  minimizes  $Z|Y$ .

(b) Let  $(x, u)$  be any feasible solution. For every  $\rho \geq \rho_{n-1}$ ,  $(x_n^*, u_n^*)$  minimizes  $Y + \rho Z$ ; hence

$$Y(x_n^*, u_n^*) + \rho Z(x_n^*, u_n^*) \leq Y(x, u) + \rho Z(x, u) \quad \text{for all } \rho \geq \rho_{n-1}$$

Rearranging, this implies

$$\rho[Z(x_n^*, u_n^*) - Z(x, u)] \leq Y(x, u) - Y(x_n^*, u_n^*) \quad \text{for all } \rho \geq \rho_{n-1}$$

This can be so only if the quantity in brackets is negative; that is, if:

$$Z(x_n^*, u_n^*) \leq Z(x, u)$$

Since this is true for any feasible  $(x, u)$ ,  $(x_n^*, u_n^*)$  minimizes  $Z$ .

Now let  $(x^*, u^*)$  be any other solution that minimizes  $Z$ . It must be true that

$$\begin{aligned} Y(x_n^*, u_n^*) + \rho_{n-1} Z(x_n^*, u_n^*) &\leq Y(x^*, u^*) + \rho_{n-1} Z(x^*, u^*) \\ Z(x_n^*, u_n^*) &= Z(x^*, u^*) \end{aligned}$$

(the former since  $(x_n^*, u_n^*)$  minimizes  $Y + \rho_{n-1} Z$ , the latter because both solutions minimize  $Z$ ). Multiplying the equation by  $\rho_{n-1}$  and subtracting from the inequality yields

$$Y(x_n^*, u_n^*) \leq Y(x^*, u^*)$$

This is true for any  $(x^*, u^*)$  minimizing  $Z$ ; hence  $(x_n^*, u_n^*)$  minimizes  $Y|Z$ .

(c) The proofs are straightforward, using the methods of (a) and (b).

Proposition 5(a) says that minimizing  $Y|Z$  yields the best solution when  $\rho = p_Z/p_Y$  is small enough. In other words, when  $p_Y$  is sufficiently large relative to  $p_Z$ ,  $Y$  dominates the total cost: the best solution is one that minimizes  $Y$  outright, then  $Z$  as much as possible. Proposition 5(b) makes the equivalent statement for the case where  $\rho = p_Z/p_Y$  is sufficiently large that  $Z$  dominates total cost.

Note that if  $n > 2$  there is at least one middle region of  $\rho$  where the best solution minimizes neither  $Y$  nor  $Z$  absolutely. When  $n = 2$ , the optimal solutions for total cost minimize either  $Y|Z$  (for  $\rho \leq \rho_1$ ) or  $Z|Y$  (for  $\rho \geq \rho_1$ ). When  $n = 1$  there is a single solution that minimizes both  $Y$  and  $Z$  absolutely, and hence minimizes any  $Y + \rho Z$ .

## §2.5 Tradeoffs between capital and operating costs

Of special interest is application of the preceding section's results to functionals  $Z_{\text{CAR}}$  and  $Z_{\text{MILE}}$ , defined in §2.1 as proportional to notions of capital cost and operating cost, respectively. Total variable cost with respect to these two objectives is

$$p_{\text{CAR}} Z_{\text{CAR}} + p_{\text{MILE}} Z_{\text{MILE}}$$

where

$$\begin{aligned} p_{\text{CAR}} &= \text{capital cost / car / day} \\ Z_{\text{CAR}} &= \text{number of cars} \\ p_{\text{MILE}} &= \text{operating cost / car-mile} \\ Z_{\text{MILE}} &= \text{car-miles / day} \end{aligned}$$

The choice of a solution that minimizes total cost depends upon  $p_{\text{CAR}}/p_{\text{MILE}}$ , the ratio of capital cost/day to operating cost/mile.

The critical ratios for this problem have a special form related to the inter-city distances, as demonstrated by the following Proposition:

Proposition 6. (a) For objectives of the form  $Z_{\text{MILE}} + (p_{\text{CAR}}/p_{\text{MILE}}) Z_{\text{CAR}}$ , the critical ratios of  $\rho = p_{\text{CAR}}/p_{\text{MILE}}$  (as defined in Proposition 4) have the form

$$\rho_i = \frac{1}{\kappa} \sum_{\substack{c, c' \in C \\ c \neq c'}} \alpha_{cc'} m_{cc'}$$

where  $\alpha_{cc'}$  are integers, and  $\kappa$  is a positive integer satisfying:

$$\kappa \leq Z_{\text{CAR}}(x_i^*, u_i^*) - Z_{\text{CAR}}(x_{i+1}^*, u_{i+1}^*)$$

(b) If the cities constitute a corridor (§1.5) ordered  $c_1, \dots, c_k$  then under the assumptions in (a) the critical ratios have the form

$$\rho_i = \frac{1}{\kappa} \sum_{\substack{c_i, c_j \in C \\ i < j}} \alpha_{c_i c_j} (m_{c_i c_j} + m_{c_j c_i})$$

where  $\alpha_{c_i c_j}$  are integers, and  $\kappa$  is a positive integer satisfying the inequality in (a).

Proof: (a) By Proposition 4, both  $(x_i^*, u_i^*)$  and  $(x_{i+1}^*, u_{i+1}^*)$  minimize  $Z_{\text{MILE}} + \rho_i Z_{\text{CAR}}$ ; that is,

$$Z_{\text{MILE}}(x_i^*, u_i^*) + \rho_i Z_{\text{CAR}}(x_i^*, u_i^*) = Z_{\text{MILE}}(x_{i+1}^*, u_{i+1}^*) + \rho_i Z_{\text{CAR}}(x_{i+1}^*, u_{i+1}^*)$$

or, by rearranging,

$$\rho_i = \frac{Z_{\text{MILE}}(x_{i+1}^*, u_{i+1}^*) - Z_{\text{MILE}}(x_i^*, u_i^*)}{Z_{\text{CAR}}(x_i^*, u_i^*) - Z_{\text{CAR}}(x_{i+1}^*, u_{i+1}^*)}$$

At any feasible solution,

- (i)  $Z_{\text{MILE}}$  is a sum of integral multiples of inter-city distances (§2.1). Hence the numerator of the above fraction is also such a sum.
- (ii)  $Z_{\text{CAR}}$  is integral. The fraction's denominator is thus a positive integer, since  $Z_{\text{CAR}}(x_{i+1}^*, u_{i+1}^*) < Z_{\text{CAR}}(x_i^*, u_i^*)$  by Proposition 4.

Ratio  $\rho_i$  thus has the desired form with  $\kappa = Z_{\text{CAR}}(x_i^*, u_i^*) - Z_{\text{CAR}}(x_{i+1}^*, u_{i+1}^*)$ .

To express  $\rho_i$  in terms of the smallest possible  $\kappa$ , divide numerator and denominator by the greatest common divisor of  $\kappa$  and all the  $\alpha_{cc'}$ .

(b) For a corridor,  $Z_{\text{MILE}}$  at any feasible solution is a sum of integral multiples of round-trip distances (§2.1). The expression for  $\rho_i$  follows from this and the arguments set forth in (a).

Proposition 6 offers a characterization of the critical ratios for  $Z_{\text{MILE}}$  and  $Z_{\text{CAR}}$ . At ratios  $\rho = p_{\text{CAR}}/p_{\text{MILE}}$  such that  $\rho_i \leq \rho < \rho_{i+1}$ , adding  $\kappa$  cars to the system makes possible a net saving of  $\sum \alpha_{cc'} m_{cc'}$  car-miles/day. So long as  $p_{\text{CAR}}/p_{\text{MILE}} > \rho_i$ , however, the Proposition implies that

$$\kappa p_{\text{CAR}} > (\sum \alpha_{cc'} m_{cc'}) p_{\text{MILE}}$$

The cost of adding  $\kappa$  cars (left-hand side) is greater than the cost saved by the reduction in car-miles (right-hand side), and hence adding the cars is uneconomical. For  $p_{\text{CAR}}/p_{\text{MILE}} < \rho_i$ , the inequality is reversed, so that total cost is less when the cars are added. When  $p_{\text{CAR}}/p_{\text{MILE}} = \rho_i$ , however,

$$\kappa p_{\text{CAR}} = (\sum \alpha_{cc'} m_{cc'}) p_{\text{MILE}}$$

Hence  $\rho_i$  is the ratio of capital to operating expense at which the capital cost of adding cars is exactly balanced by a resultant saving in operating cost.

For the Northeast Corridor data described later in this report, all critical ratios had the especially simple form  $\kappa = 1$ ,  $\alpha_{c_j c_j} = 0$  or 1. At these ratios the capital cost of one added car equalled the operating cost over one round trip that was saved by adding the car: see further in §§4.4, 4.6. (It may be that under certain assumptions about the network and schedule, critical ratios must have simple forms like this; but this has not been shown formally.)

## PART B

### APPLICATIONS TO THE NORTHEAST CORRIDOR

The remainder of this report describes how an instance of the model of §1 and §2 can be used to investigate rail service in the Northeast Corridor.

For purposes of demonstration, a hypothetical case representing service on a busy day in 1982 was chosen as a basis for analysis. *Base data* for this case were estimated by the means described in §3. These data were incorporated in an appropriate network model, which was solved and analyzed by use of the SESAME interactive linear programming system [3,6,7] and supporting computer routines. Details of this *base run*, and the numerical results, are given in §4.

The base run is not intended as a thorough analysis of 1982 Corridor service, but as a test case that prepares the way for further analyses. Compilation of the base data, for example, has led to development of techniques that are now available for more extensive studies. Output from the base run, moreover, has revealed some special properties of the Corridor network, which in turn might be exploited in subsequent models (see, for instance, §4.5).

In addition, application of the model requires an integrated set of interactive computer routines. These were developed and tested for the base run and are now available to other users via a time-sharing network. Instructions for use of the computer routines are provided in Volume II of this report [4].



### §3 THE BASE DATA

This section describes how an instance of the network model was formulated for the Northeast Corridor (§§3.1-3.2), and how base data were derived for the model (§§3.3-3.8).

The primary reference for data-gathering techniques is a pair of Corridor studies prepared by Peat, Marwick, Mitchell and Company [1,5]. These are referred to in the sequel as the "PMM studies".

The techniques for computing demands described in §§3.4-3.7 were developed by Judith Gertler and Howard Sinkowitz of the Transportation Systems Center, U.S. Department of Transportation.

#### §3.1 Characteristics of the Northeast Corridor

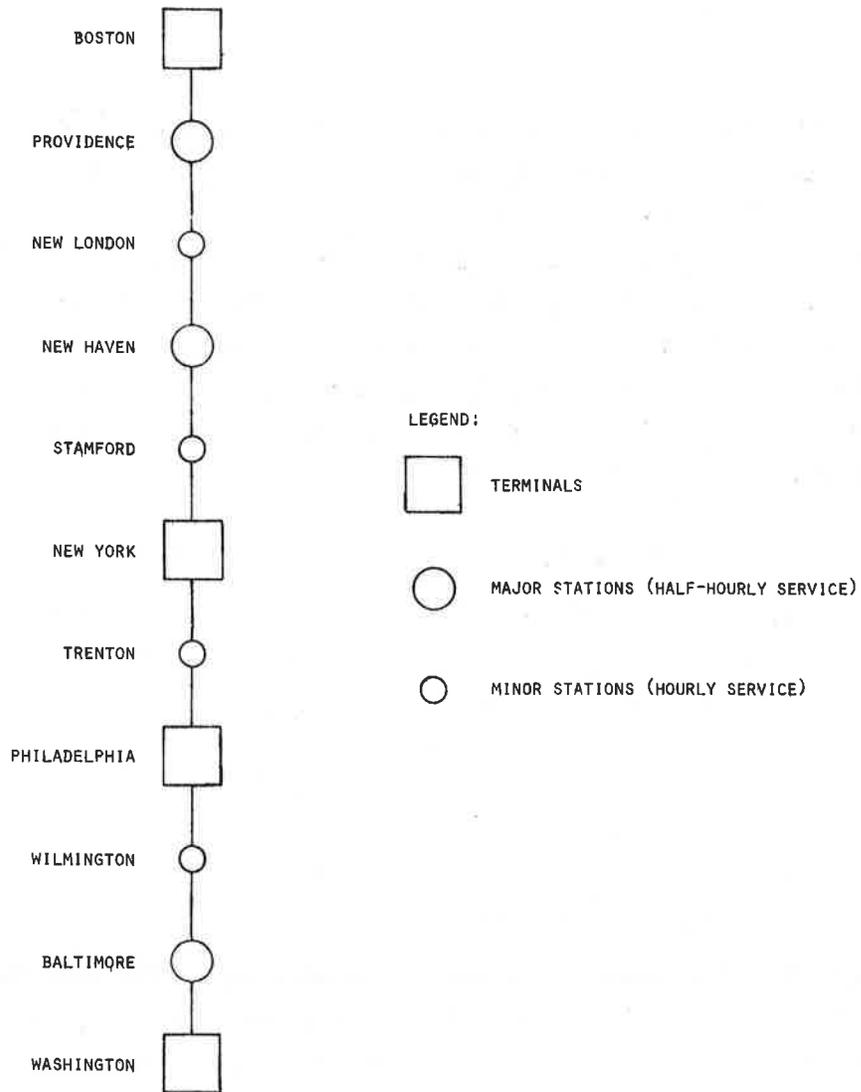
The base-run Northeast Corridor comprises four *terminals*: Boston, New York, Philadelphia, and Washington. Scheduled trains connect these terminals on three north-south *segments* as follows:

<u>Segment</u>	<u>Length (miles)</u>
Boston - New York	232
New York - Philadelphia	90
Philadelphia - Washington	135

Cars arriving at a terminal may move on immediately in the same direction, or may be stored for use in later trains in either direction. A fixed minimum amount of time (in addition to the normal stopover time) is required to change the direction of a car.

Also in the Corridor are seven intermediate *stations*, as shown in Figure 3-1. Trains are scheduled to stop at these stations, but cars may not be stored or switched there. Including both terminals and intermediate stations, the corridor comprises 11 cities, connected by 10 north-south *links*.

Figure 3-1. Terminals and intermediate stations in the Northeast Corridor as modeled by the base run.



For purposes of the 1982 base run, cars in Corridor service are assumed to have a uniform capacity of 75 passengers. Station size is taken to be 14 cars; trains requiring more than 14 cars are to be run in multiple sections of 14 cars or less each. Each section is assumed to require one locomotive.

### §3.2 Modeling Northeast Corridor service

The Northeast Corridor is modeled as a corridor network with turnaround delay, as defined in §1.5.

The set  $C$  of cities in the model comprises the four Corridor terminals:

$$C = \{\text{Boston, New York, Philadelphia, Washington}\}$$

Intermediate stations can be omitted from this set, since they are not points at which cars may be stored or switched. (Demands to and from intermediate stations are used to determine the minimum size of each train, however. See §§3.4-3.6).

The model's schedule-period is one day, partitioned into a set of intervals  $T$  representing minutes of the day. Hence the number of partition intervals,  $\tau$ , is 1440.

The schedule,  $S$ , lists the arrival and departure terminal of each train and the corresponding arrival and departure times to the nearest minute. Its construction is described in §3.3.

The demand for each train is calculated from annual patronage forecasts by the methods described in §§3.4-3.6. A lower limit  $d_{cc}, [t, t]$  and upper limit  $h_{cc}, [t, t']$  on each train's size is then derived from its demand, as explained by §3.7.

The turnaround delay  $\delta$  is fixed at 20 minutes, for reasons set forth in §3.8.

### §3.3 The schedule

The 11-city base schedule (Figure 3-2) is an updated version of that employed in the PMM studies [1,5]. It assumes generally half-hourly service to the terminals and major intermediate stations (Providence, New Haven, Baltimore), and hourly service at minor stations (New London, Stamford, Trenton, Wilmington).

Figure 3-2(a). Schedule of northbound trains employed in the base run.

<u>Line</u>	<u>WASH</u>	<u>BALT</u>	<u>WILM</u>	<u>PHIL</u>	<u>TREN</u>	<u>NYC</u>	<u>STAM</u>	<u>NEW H</u>	<u>NEW L</u>	<u>PROV</u>	<u>BOST</u>
2						4:30	---	5:53	---	7:30	8:10
4	2:45	3:21	4:05	4:24	4:47	5:30	6:35	6:53	7:45	8:30	9:10
6	3:45	4:21	5:05	5:24	5:47	6:30	7:35	7:53	8:45	9:30	10:10
8						7:00	---	8:23	---	10:00	10:40
10	4:45	5:21	6:05	6:24	6:47	7:30	8:35	8:53	9:45	10:30	11:10
12				6:54	---	8:00	---	9:23	---	11:00	11:40
14	5:45	6:21	7:05	7:24	7:47	8:30	9:35	9:53	10:45	11:30	12:10
16	6:15	6:51	---	7:54	---	9:00	---	10:23	---	12:00	12:40
18	6:45	7:21	8:05	8:24	8:47	9:30	10:35	10:53	11:45	12:30	13:10
20	7:15	7:51	---	8:54	---	10:00	---	11:23	---	13:00	13:40
22	7:45	8:21	9:05	9:24	9:47	10:30	11:35	11:53	12:45	13:30	14:10
24	8:15	8:51	---	9:54	---	11:00	---	12:23	---	14:00	14:40
26	8:45	9:21	10:05	10:24	10:47	11:30	12:35	12:53	13:45	14:30	15:10
28	9:15	9:51	---	10:54	---	12:00	---	13:23	---	15:00	15:40
30	9:45	10:21	11:05	11:24	11:47	12:30	13:35	13:53	14:45	15:30	16:10
32	10:15	10:51	---	11:54	---	13:00	---	14:23	---	16:00	16:40
34	10:45	11:21	12:05	12:24	12:47	13:30	14:35	14:53	15:45	16:30	17:10
36	11:15	11:51	---	12:54	---	14:00	---	15:23	---	17:00	17:40
38	11:45	12:21	13:05	13:24	13:47	14:30	15:35	15:53	16:45	17:30	18:10
40	12:15	12:51	---	13:54	---	15:00	---	16:23	---	18:00	18:40
42	12:45	13:21	14:05	14:24	14:47	15:30	16:35	16:53	17:45	18:30	19:10
44	13:15	13:51	---	14:54	---	16:00	---	17:23	---	19:00	19:40
46	13:45	14:21	15:05	15:24	15:47	16:30	17:35	17:53	18:45	19:30	20:10
48	14:15	14:51	---	15:54	---	17:00	---	18:23	---	20:00	20:40
50	14:45	15:21	16:05	16:24	16:47	17:30	18:35	18:53	19:45	20:30	21:10
52	15:15	15:51	---	16:54	---	18:00	---	19:23	---	21:00	21:40
54	15:45	16:21	17:05	17:24	17:47	18:30	19:35	19:53	20:45	21:30	22:10
56	16:15	16:51	---	17:54	---	19:00	---	20:23	---	22:00	22:40
58	16:45	17:21	18:05	18:24	18:47	19:30	20:35	20:53	21:45	22:30	23:10
60	17:15	17:51	---	18:54	---	20:00	---	21:23	---	23:00	23:40
62	17:45	18:21	19:05	19:24	19:47	20:30	21:35	21:53	22:45	23:30	24:10
64	18:15	18:51	---	19:54	---	21:00	---	22:23	---	24:00	24:40
66	18:45	19:21	20:05	20:24	20:47	21:30	22:35	22:53	23:45	24:30	1:10
68	19:15	19:51	---	20:54	---	22:00	---	23:23	---	1:00	1:40
70	19:45	20:21	21:05	21:24	21:47	22:30	23:35	23:53	24:45	1:30	2:10
72	20:15	20:51	---	21:54	---	23:00	---	---	---	---	---
74	20:45	21:21	22:05	22:24	22:47	23:30	24:35	24:53	1:45	2:30	3:10
76	21:45	22:21	---	23:24	---	24:30	---	---	---	---	---
78	22:45	23:21	---	24:24	---	1:30	---	---	---	---	---
80	24:45	1:21	2:05	2:24	2:47	3:30	---	---	---	---	---

Figure 3-2(b). Schedule of southbound trains employed in the base run.

Line	BOST	PROV	NEW L	NEW H	STAM	NYC	TREN	PHIL	WILM	BALT	WASH
1						5:30	---	6:33	---	7:35	8:10
3						6:00	6:39	7:03	7:22	8:05	8:40
5						6:30	---	7:33	---	8:35	9:10
7	3:15	3:56	4:42	5:33	5:51	7:00	7:39	8:03	8:22	9:05	9:40
9						7:30	---	8:33	---	9:35	10:10
11	4:15	4:56	5:42	6:33	6:51	8:00	8:39	9:03	9:22	10:05	10:40
13						8:30	---	9:33	---	10:35	11:10
15	5:15	5:56	6:42	7:33	7:51	9:00	9:39	10:03	10:22	11:05	11:40
17	5:45	6:26	---	8:03	---	9:30	---	10:33	---	11:35	12:10
19	6:15	6:56	7:42	8:33	8:51	10:00	10:39	11:03	11:22	12:05	12:40
21	6:45	7:26	---	9:03	---	10:30	---	11:33	---	12:35	13:10
23	7:15	7:56	8:42	9:33	9:51	11:00	11:39	12:03	12:22	13:05	13:40
25	7:45	8:26	---	10:03	---	11:30	---	12:33	---	13:35	14:10
27	8:15	8:56	9:42	10:33	10:51	12:00	12:39	13:03	13:22	14:05	14:40
29	8:45	9:26	---	11:03	---	12:30	---	13:33	---	14:35	15:10
31	9:15	9:56	10:42	11:33	11:51	13:00	13:39	14:03	14:22	15:05	15:40
33	9:45	10:26	---	12:03	---	13:30	---	14:33	---	15:35	16:10
35	10:15	10:56	11:42	12:33	12:51	14:00	14:39	15:03	15:22	16:05	16:40
37	10:45	11:26	---	13:03	---	14:30	---	15:33	---	16:35	17:10
39	11:15	11:56	12:42	13:33	13:51	15:00	15:39	16:03	16:22	17:05	17:40
41	11:45	12:26	---	14:03	---	15:30	---	16:33	---	17:35	18:10
43	12:15	12:56	13:42	14:33	14:51	16:00	16:39	17:03	17:22	18:05	18:40
45	12:45	13:26	---	15:03	---	16:30	---	17:33	---	18:35	19:10
47	13:15	13:56	14:42	15:33	15:51	17:00	17:39	18:03	18:22	19:05	19:40
49	13:45	14:26	---	16:03	---	17:30	---	18:33	---	19:35	20:10
51	14:15	14:56	15:42	16:33	16:51	18:00	18:39	19:03	19:22	20:05	20:40
53	14:45	15:26	---	17:03	---	18:30	---	19:33	---	20:35	21:10
55	15:15	15:56	16:42	17:33	17:51	19:00	19:39	20:03	20:22	21:05	21:40
57	15:45	16:26	---	18:03	---	19:30	---	20:33	---	21:35	22:10
59	16:15	16:56	17:42	18:33	18:51	20:00	20:39	21:03	21:22	22:05	22:40
61	16:45	17:26	---	19:03	---	20:30	---	21:33	---	22:35	23:10
63	17:15	17:56	18:42	19:33	19:51	21:00	21:39	22:03	22:22	23:05	23:40
65	17:45	18:26	---	20:03	---	21:30	---	22:33	---	23:35	24:10
67	18:15	18:56	19:42	20:33	20:51	22:00	22:39	23:03	23:22	24:05	24:40
69	18:45	19:26	---	21:03	---	22:30	---	23:33	---	---	---
71	19:15	19:56	20:42	21:33	21:51	23:00	23:39	24:03	24:22	1:05	1:40
73	19:45	20:26	---	22:03	---	23:30	---	---	---	---	---
75	20:15	20:56	21:42	22:33	22:51	24:00	24:39	1:03	---	---	---
77	21:15	21:56	22:42	23:33	23:51	1:00	---	---	---	---	---
79	22:15	22:56	23:42	24:33	24:51	2:00	---	---	---	---	---
81	23:15	23:56	24:42	1:33	1:51	3:00	---	---	---	---	---

Appropriate reductions are made late at night and early in the morning, when demands are very low.

Segment trip times for 1982 are assumed to be approximately as follows:

<u>Segment</u>	<u>Trip time</u>
Boston - New York	3 hours 40 minutes
New York - Philadelphia	1 hour 1 minute
Philadelphia - Washington	1 hour 38 minutes

Trip times for individual links are calculated accordingly. Allowance is made in addition for stopover times of about 5 minutes at New York, and 1.25 minutes at other stations. It is assumed, however, that trains do not save any time when they skip stops at minor stations.

The schedule is centered around New York, where arrivals and departures occur on the half-hour. The patterns of activity at other terminals vary, depending on the trip times from New York.

A four-city schedule, incorporating just the terminals, is abstracted from the base schedule for use by the model. Each trip over a segment is modeled as an independent train: line 19 of the schedule, for example, is treated as a Boston-New York train, followed by a New York-Philadelphia train, followed by a Philadelphia-Washington train. These three trains most likely employ the same locomotive, however, and many of the same cars, so that from the passenger's point of view there would be through service from Boston all the way to Washington. The same is true for other south-bound trains, and the reverse for northbound trains, with a few exceptions at extreme hours.

#### §3.4 Design-day patronage

Annual patronage for 1982 was calculated by use of a computer-based model developed in one of the PMM studies [5]. The input data were those derived from PMM's "base assumptions" with the exception of trip times, which were increased to reflect the base schedule (§3.3).

The PMM model estimates annual two-way patronage for individual station-pairs in the Northeast Corridor, as shown in Figure 3-3. Annual one-way patronage is computed by halving the two-way figures. A few

Figure 3-3. Projected 1982 rail patronage for station-pairs in the Northeast Corridor, as calculated for the base run.

	BOST	PROV	NEW L	NEW H	STAM	NYC	TREN	PHIL	WILM	BALT
PROV	530997									
NEW L	102153	18449								
NEW H	128506	29931	34785							
STAM	394556	73273	120135	---						
NYC	1242149	279799	210230	613866	---					
TREN	8696	3827	855	3037	4710	1812035				
PHIL	109390	26008	12833	12103	52346	4827117	---			
WILM	3752	1972	694	4533	7269	369249	4736	197841		
BALT	21258	7100	2095	10592	19655	720045	19241	334283	34979	
WASH	81386	16469	13847	30647	190100	1722566	94489	941846	106030	231422

possible station-pairs are omitted, either because they could not be separated from other pairs, or because competitive commuter service is available for their travelers. All of these excluded pairs are short distance, and are deemed to be relatively insignificant to Corridor service.

The base run models patronage for a *design day*, calculated as 1/270 of the annual amount. This concept of design day, representing approximately the 10th busiest day of the year, has been employed in engineering studies of the Northeast Corridor [2, pp. 3-35] and in fleet-sizing experiments [1, Appendix C].

### §3.5 Demand distributions

The base run employs a set of cumulative demand functions to derive the patterns of demand between station-pairs over a day. Following a method of the PMM studies [1, pp. C.7-C.14], demand for service from a larger station to a smaller one is taken to be departure-based (that is, dependent upon the time of departure), while demand for service from a smaller to a larger station is arrival-based (dependent upon time of arrival). Demand between cities of comparable size is determined by averaging arrival-based and departure-based distribution functions.

The demand distributions employed in the base run are derived from bi-modal gaussian-like probability distributions\* fit to actual arrival and departure counts for Tuesday, May 21, 1974. This date was chosen because it afforded actual ticketing data, and was uninfluenced by special weekend or holiday patterns. Counts could be made, however, for only a small number of station-pairs, especially as no information was available for trips passing through New York. In consequence, actual distributions were fit for the following pairs only:

<u>Station-pair</u>	<u>Based on:</u>
Washington - New York	arrival
New York - Washington	departure
Philadelphia - New York	arrival
New York - Philadelphia	departure
Washington - Philadelphia	arrival

---

\*These distributions were derived and estimated by Walter Messcher and Alan Wellington at the Transportation Systems Center, U.S. Department of Transportation. See further in the Appendix to this volume.

Figure 3-4. Cumulative distributions employed in calculating demands. The entry in each box indicates a distribution (fit to actual data) used to estimate demand for all trains from the associated departure city (left margin) to the associated arrival city (top margin). "Arrival" indicates an arrival-based distribution, "departure" a departure-based distribution, and "arr + dep" an average of an arrival-based distribution and a departure-based one.

		T O :										
		BOSTON	PROVIDENCE	NEW LONDON	NEW HAVEN	STAMFORD	NEW YORK	TRENTON	PHILADELPHIA	WILMINGTON	BALTIMORE	WASHINGTON
FROM :	BOSTON		New York - Phila. departure	New York - Phila. departure	New York - Wash. departure	New York - Wash. departure	Wash. - New York arrival	New York - Wash. departure	Wash. - Phila. arr + dep	Wash. - Phila. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arrival
	PROVIDENCE	Phila. - New York arrival		Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - New York arrival	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arrival
	NEW LONDON	Phila. - New York arrival	Wash. - Wilm. arr + dep		Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Phila. - New York arrival	Wash. - Wilm. arr + dep	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - Wilm. arrival	Wash. - Wilm. arrival
	NEW HAVEN	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep			Phila. - New York arrival	Wash. - Wilm. arr + dep	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - Wilm. arrival	Wash. - Wilm. arrival
	STAMFORD	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep				Wash. - Wilm. arr + dep	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Wash. - Wilm. arrival
	NEW YORK	New York - Wash. departure	New York - Wash. departure	New York - Phila. departure	New York - Phila. departure			New York - Phila. departure	New York - Phila. departure	New York - Phila. departure	New York - Wash. departure	New York - Wash. departure
	TRENTON	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Phila. - New York arrival			Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep
	PHILADELPHIA	Phila. - Wash. arr + dep	New York - Wash. departure	New York - Wash. departure	New York - Wash. departure	New York - Wash. departure	Phila. - New York arrival			New York - Phila. departure	Phila. - Wash. departure	Phila. - Wash. departure
	WILMINGTON	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep	Phila. - New York arrival	Wash. - Wilm. arr + dep	Phila. - New York arrival		Wash. - Wilm. arr + dep	Wash. - Wilm. arr + dep
	BALTIMORE	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arr + dep	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - Phila. arrival	Wash. - Wilm. arr + dep		Phila. - New York arrival
	WASHINGTON	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - Wilm. arrival	Wash. - New York arrival	Wash. - Wilm. arr + dep	Wash. - Phila. arrival	Wash. - Wilm. arr + dep	New York - Phila. departure	

<u>Station-pair</u>	<u>Based on:</u>
Washington - Philadelphia	departure
Philadelphia - Washington	arrival
Philadelphia - Washington	departure
Washington - Wilmington	arrival
Washington - Wilmington	departure

For other station-pairs, the actual distribution is approximated by an appropriate distribution chosen from this list: see Figure 3-4.

### §3.6 Effective demands over segments

For every scheduled train over a segment, there is an *effective demand*: the number of passengers that the train must accommodate to guarantee everyone a seat at all points on the route. Effective demands are determined for the base run in the following steps:

Station-pair demands. Given one-way patronage data (§3.4) and cumulative demand functions (§3.5), a design-day demand is calculated for every scheduled trip between a pair of stations in the 11-city schedule (excluding certain station-pairs as explained in §3.3).

Link demands. The total demand for a train over a given link is computed. It equals the sum of demands for all station-pair trips that employ the train over that link.

For example, total demand for a typical scheduled train over the Wilmington-Philadelphia link does not include just passengers who get on at Wilmington and disembark at Philadelphia. Some passengers who get off at Philadelphia began their trip in Washington or Baltimore; some who start at Wilmington will stay on to Trenton, New York, or a station further north; and some passengers both start south of Wilmington and terminate north of Philadelphia. Demand for the train for all such station-pair trips must be added to determine total demand for the train over the Wilmington-Philadelphia link.

Maximal-link demands. For every train over a particular segment, there emerges from the link demands a *maximal link* over which demand is highest. A

train accommodates all passengers over a segment only if it meets demand over the maximal link, since cars cannot be added within the segment. Hence the effective demand for each train is equal to the train's maximal link demand.

For instance, say demand for a Washington-Philadelphia train is 197 passengers over the Washington-Baltimore link, 237 over the Baltimore-Wilmington link, and 225 over the Wilmington-Philadelphia link. The maximal link for that train is then Baltimore-Wilmington, and effective demand for the train is 237.

### §3.7 Minimum and maximum train sizes

For the base run, cars are assumed to hold 75 passengers. Hence if  $d$  is the effective demand for a train, its minimum size is:

$$\langle d/75 \rangle$$

(Here angle brackets denote the least integer greater than the enclosed value.)

The maximum size of a train for the base run is the lesser of two limits, one related to load factor, the other to station length:

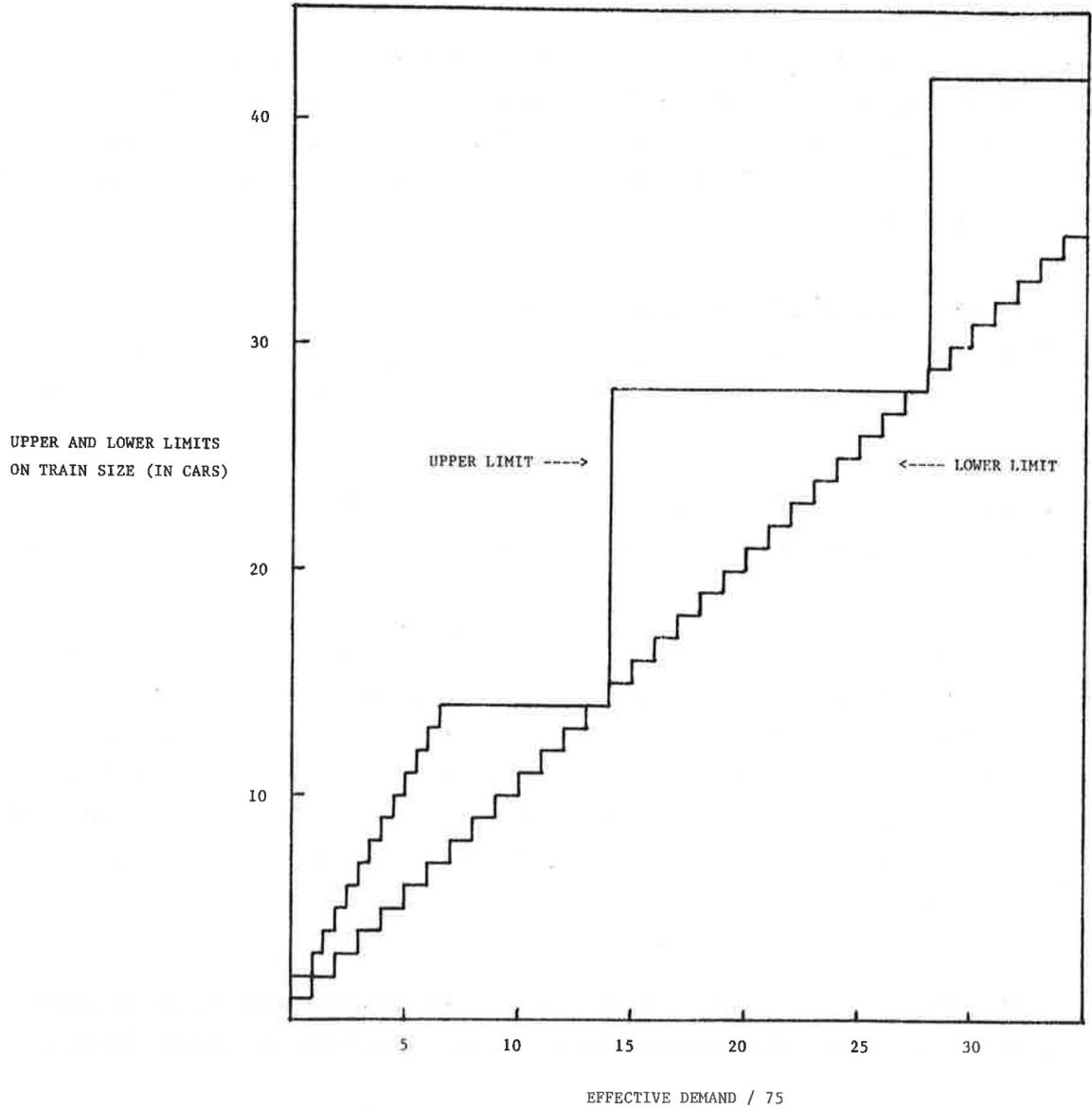
Load-factor limit. Due to imbalances in demand throughout the day, some trains will have to be run with more than the absolute minimum number of cars. It is reasonable, however, to limit the number of these deadhead cars to some proportion of the train. Specifically, in the base run no train is allowed to have more cars than required to meet twice its effective demand, with the proviso that every train may have at least 2 cars. In terms of  $d$ , this limit is:

$$\max (\langle 2d/75 \rangle, 2)$$

Its effect is to require load factor over the maximal link to be at least reasonably near 50%, the requirement becoming stricter at larger demands.

Station-length limit. Plans for 1982 assume that stations will hold at most 14-car trains (§3.1). When more than 14 cars are assigned to a train, one or more extra sections must be put on, employing an equal number

Figure 3-5. Train-size limits for the base run as a function of effective demand.



of extra locomotives. To prevent unnecessary extra sections, the base run requires that the number of sections actually run be no greater than  $\langle(d/75)/14\rangle$ , the number of sections required to meet effective demand. This translates to an upper limit on cars of:

$$14 \langle(d/75)/14\rangle$$

If  $d/75$  is 12.6, for instance, this upper limit is 14; but if  $d/75$  is 15.2, two sections are needed in any event, and the limit is 28.

Figure 3-5 graphs train-size limits as a function of effective demand. The load-factor upper limit is the lesser one for demands under 525 passengers (7 cars). At larger demands, the station-length limit predominates.

For the base run, only 5 trains require as many as two sections: three from Philadelphia to New York in the morning, and two from New York to Philadelphia in the afternoon. Most other trains of 7 or more cars are also on the New York-Philadelphia segment.

### §3.8 Turnaround delay

For the base run a delay of 20 minutes (in addition to the regular stopping time built into the schedule) is postulated whenever the direction of a car is changed. This time is deemed sufficient to cover switching under 1982 conditions plus any lags in the arrival of extra sections.

A look at the schedule shows that any turnaround delay from 9 to 20 minutes has the same effect, while a delay of more than 20 minutes requires additional cars at Philadelphia.



Computer processing and its results for the base run are discussed in this section.

Generation of the base data (§4.1) and the LP model (§4.2) were necessarily performed first. Optimal solutions were then found for a variety of objectives: minimum cars and car-miles, and maximum load factor (§4.3); minimum total operating and capital cost (§4.4); and minimum turnaround switching (§4.5). Further analyses included sensitivity to demands (§4.6) and requirements for locomotives (§4.7).

The principal computing tool was the SESAME linear programming system developed at the National Bureau of Economic Research [3,6,7]. SESAME, and the other computer routines employed in this study, have been made available for use through a national time-sharing network. Prospective users should consult Volume 2 of this report [4].

#### §4.1 Computing the base data

Estimates of 1982 rail patronage (§3.4) were produced by running a computer simulation program adapted specially for the PMM demand study [5]. This program projects business and non-business use of four modes of travel: rail, bus, air, and car. A subroutine was added to file total rail patronage only, in a format suitable for subsequent processing.

The patronage data file, plus a file representing the full schedule, then served as input to a demand program written by Judith Gertler and Howard Sinkowitz of the Transportation Systems Center. This program employs cumulative demand functions for station-pairs to compute effective demands, and consequent upper and lower limits, for each train (as described in §§3.5-3.7).

Principal output from the demand program is a set of tables, representing the schedule and other information, that can be read by an LP matrix generator (§4.2). In addition, sets of alternative train-size limits are filed in a form that allows any one set to be read into the matrix.

#### §4.2 Generating the model

An LP equivalent of the network model was generated in a form suitable for computer processing by DATAMAT, a subsystem of SESAME [3]. A program in the DATAMAT macro language was written for this purpose. Its inputs include the schedule and other tables produced from the base data (§4.1), and the turn-around delay in minutes. Output is an arrangement of LP rows and columns in an internal form suitable for processing by SESAME.

Upper and lower limits on train-size variables are not generated as explicit constraint rows; they are incorporated in a "bound set" that is enforced implicitly by SESAME's simplex algorithm. Actual limit values are also absent at this stage: they are read in from a separate file just before the model is solved or analyzed. This arrangement facilitates working with several sets of limits, as was done, for example, in the sensitivity analysis described in §4.6.

The LP generated by the DATAMAT program represents a reduced network, duplicated to distinguish northbound and southbound cars in the corridor (§§1.3, 1.5). For the base schedule, the LP representation required 1275 structural variables and 528 constraint rows.

#### §4.3 Minimizing cars and car-miles

The base-run LP was solved by use of SESAME's standard primal simplex algorithm. A feasible solution was obtained (starting from an all-slack basis) in 665 iterations, and an optimal solution for the minimum-cars objective,  $Z_{CAR}$ , in an additional 28 iterations. An optimal solution for the minimum-car-miles objective,  $Z_{MILE}$ , was also found. A maximum system load factor,  $Z_{LF}$ , was determined from  $Z_{MILE}$ , as these two objectives are inversely proportional (§2.1).

The values of the objectives at their optima for the base data are:

$$\begin{aligned}\min Z_{CAR} &= 164 \text{ cars} \\ \min Z_{MILE} &= 131388 \text{ car-miles} \\ \max Z_{LF} &= 74.15\%\end{aligned}$$

#### §4.4 Minimizing operating plus capital cost

Following the analysis set forth in §§2.3-2.5, the next step was to minimize total "operating" and "capital" cost of the base model, expressed as:

$$p_{\text{CAR}} z_{\text{CAR}} + p_{\text{MILE}} z_{\text{MILE}}$$

where:

$$\begin{aligned} p_{\text{CAR}} &= \text{capital cost / car / day} \\ z_{\text{CAR}} &= \text{number of cars in the system} \\ p_{\text{MILE}} &= \text{operating cost / car-mile} \\ z_{\text{MILE}} &= \text{car-miles / day} \end{aligned}$$

SESAME's algorithm for parametric analysis of the objective function was employed for this purpose. Part of the process was automated by use of small programs written in the SESAME command language.

The properties of an optimal solution depend on the value of  $p_{\text{CAR}}/p_{\text{MILE}}$ , the ratio of capital cost/day to operating cost/mile. For the base data, there are three significantly different regions into which this ratio may fall:

(1) Capital cost/day  $\geq$  450 · operating cost/mile. Here capital cost dominates; in any optimal solution the number of cars is at its absolute minimum, 164. The minimum number of car-miles per day, given 164 cars, is 135978; and the system load factor (which is inversely proportional to total car-miles) is 71.65%.

(2) 450 · operating cost/mile  $\geq$  capital cost/day  $\geq$  180 · operating cost/mile. At this level the influence of capital cost declines somewhat. The number of cars in an optimal solution increases to 167; car-miles per day decline to 134628 (system load factor 72.37%).

(3) Capital cost/day  $\leq$  180 · operating cost/mile. Here operating cost dominates. In an optimal solution car-miles/day is at its absolute minimum, 131388 (system load factor 74.15%), while the number of cars in the system increases to 185.

Figure 4-1. Cars and car-miles when total cost is minimized, as a function of the ratio of capital cost/day to operating cost/mile. The critical ratios are 180 and 450.

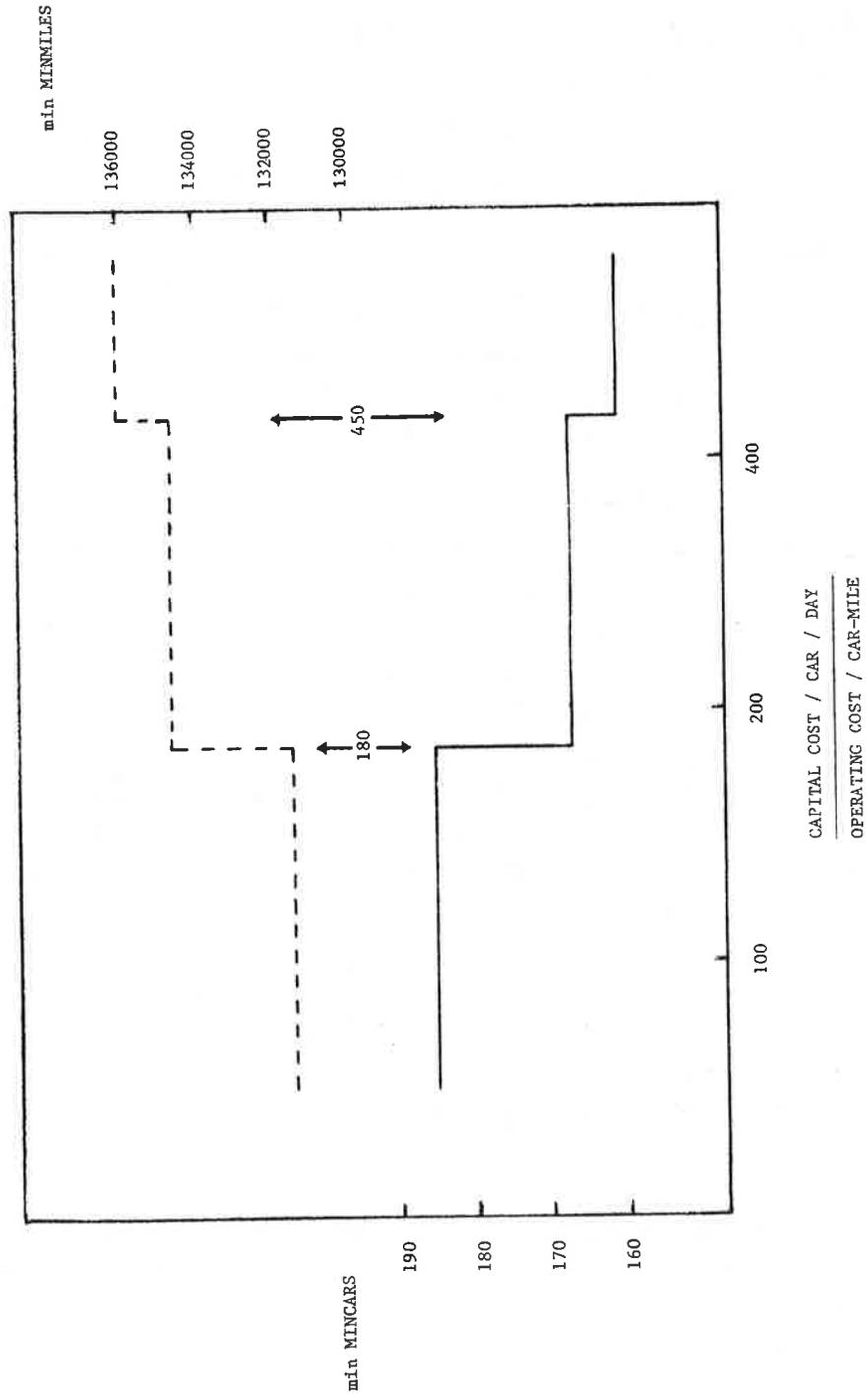
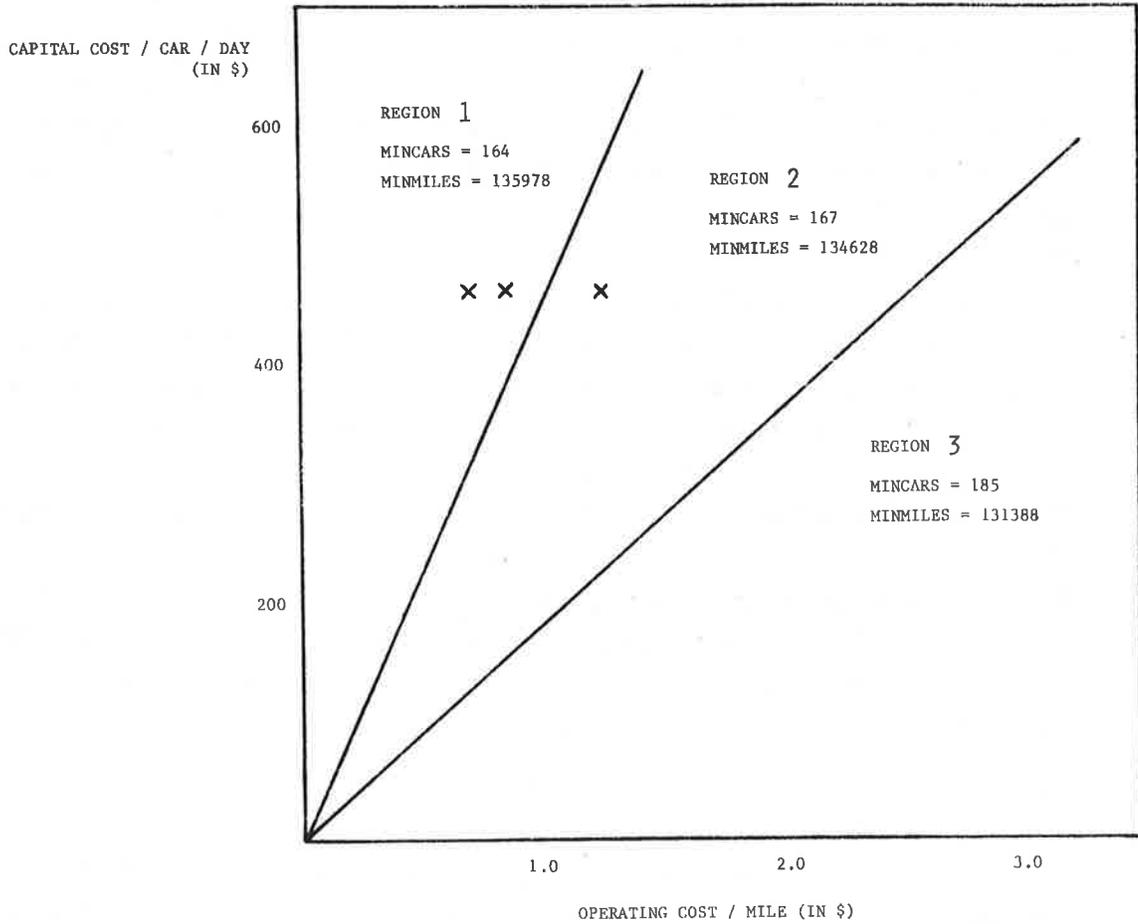


Figure 4-2. The three solution regions for the base run, plotted as a function of capital cost/day and operating cost/mile. Alternative projections of the actual ratio of these amounts in 1982 are indicated by X's.



The results are shown graphically in Figure 4-1. Clearly the biggest jump is at critical ratio  $p_{\text{CAR}}/p_{\text{MILE}} = 180$ , the round-trip distance between New York and Philadelphia. At ratios below this point, buying an extra car is economical even if it saves just one New York-Philadelphia run. At higher ratios it pays to buy a smaller fleet, running each car (on the average) more miles every day. The size of the jump -- about a 10% difference in fleet size -- is not surprising. Demand is heaviest along the New York-Philadelphia segment, and is highly unbalanced: northbound travel peaks in the morning, while southbound demand is highest in the afternoon. Consequently, a fair amount of deadheading can be avoided if a larger fleet is available.

The other jump, at  $p_{\text{CAR}}/p_{\text{MILE}} = 450$ , represents a point at which the cost of a car equals the cost of running it from New York to Washington and back. This is a fairly insignificant critical ratio, however, as the optimum at ratios below 450 requires only three cars more than the optimum above 450.

Several estimates of the actual  $p_{\text{CAR}}/p_{\text{MILE}}$  were derived from a PMM financial analysis [1] by Judith Gertler and Howard Simkowitz of the Transportation Systems Center. These are plotted against the critical ratios in Figure 4-2. The estimates suggest that  $p_{\text{CAR}}/p_{\text{MILE}}$  probably falls into region (1), and hence that capital cost is probably predominant. (Moreover, if the ratio is not in region (1) it appears very likely to be in region (2), where the optimal solution is much different.)

#### §4.5 Minimizing turnaround

We next looked at a third objective,  $Z_{\text{TURN}}$ , the number of times per day that the direction of a car is changed (§2.1). Since capital cost seemed likely to predominate, we found a solution to:

$$\min Z_{\text{TURN}} | Z_{\text{MILE}} | Z_{\text{CAR}}$$

That is,  $Z_{\text{CAR}}$  was minimized first (164 cars); then  $Z_{\text{MILE}}$  was minimized subject to  $Z_{\text{CAR}} = 164$  (135978 car-miles); then  $Z_{\text{TURN}}$  was minimized subject to  $Z_{\text{MILE}} = 164$  and  $Z_{\text{CAR}} = 135978$ .

The optimal value of  $Z_{\text{TURN}}$  is not particularly revealing; but the flow of cars being turned around at New York and Philadelphia is of interest. No northbound car is ever turned around at Philadelphia, and no southbound car is ever turned at New York. Cars running north from Philadelphia are held in storage at New York mostly in the morning, when northbound travel on the Philadelphia-New York segment predominates. Cars running south from New York are held at Philadelphia mostly in the afternoon, when southbound traffic is dominant on the segment.

In effect, many cars are needed only for the Philadelphia-New York segment, to satisfy peak demand northbound in the morning and southbound in the afternoon. This suggests a revised schedule in which New York-Philadelphia shuttle trains are run at peak hours, in addition to the usual through trains.

#### §4.6 Sensitivity to demand

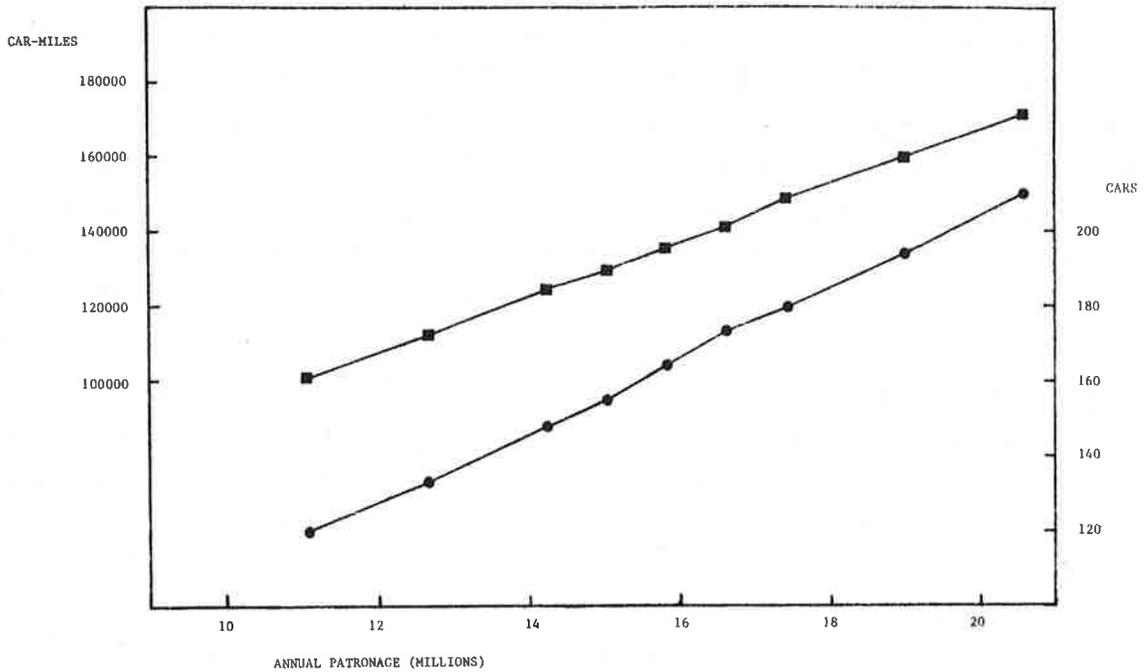
Demand projections are inherently uncertain. They are based on approximate data, and their postulations are open to question. A PMM study of Northeast Corridor demands [5], for example, estimates 1982 patronage at anywhere from 11 to 23 million passengers, depending upon assumptions about costs and travel times.

It is thus essential that the model be solved for a range of demands. Fortunately, this can be done by SESAME in an especially efficient way, by taking advantage of two characteristics of the model.

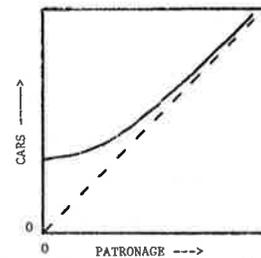
First, a change in demands does not change the model's row and column structure: demands affect only the lower and upper limits on the train-size variables. Consequently, the LP matrix need be generated only once for each combination of schedule and turnaround delay. Sets of limit values are filed separately; just before the model is to be solved or analyzed, SESAME is instructed which set of limits to use with the previously-created matrix. Any different set of limits is easily substituted whenever desired.

Second, different sets of demand limits for the same model tend to be similar, and hence their optimal solutions are generally close together. As a result, it is not necessary to solve from scratch for each set of demands. An optimal basis for one demand set is a very good starting basis for iterating to an optimum for any similar set. SESAME's dual simplex algorithm is especially useful for this purpose, since changing upper and lower limits does not violate dual feasibility.

Figure 4-3.  $\min Z_{CAR}$  (right scale) and  $\min Z_{MILE} / Z_{CAR}$  (left scale) as a function of annual patronage. The small graph shows the general forms of these functions as patronage approaches zero and infinity.



TO RIGHT:  
ASYMPTOTIC BEHAVIOR OF CARS (OR CAR-MILES) vs. PATRONAGE



For the base run, alternative estimates of effective demands were first derived through scaling the base patronage estimates by a constant factor; then upper and lower limits were determined as before. Nine factors, ranging from .7 to 1.3, were chosen. For each, a separate set of upper and lower limits was filed by the demand program (§4.1).

An analysis of total capital and operating cost was performed, in the manner of §4.4, for each set of scaled demands. The overall pattern is the same as that for the base demands: the only large jump is at  $p_{\text{CAR}}/p_{\text{MILE}} = 180$ , where the capital cost of a car equals its operating cost from New York to Philadelphia and back. There is some variation in the minor jumps, the one at 450 (New York-Washington) sometimes omitted, and one at 462 (New York-Boston) occasionally appearing; but none of these jumps is associated with a significant change in the solution.

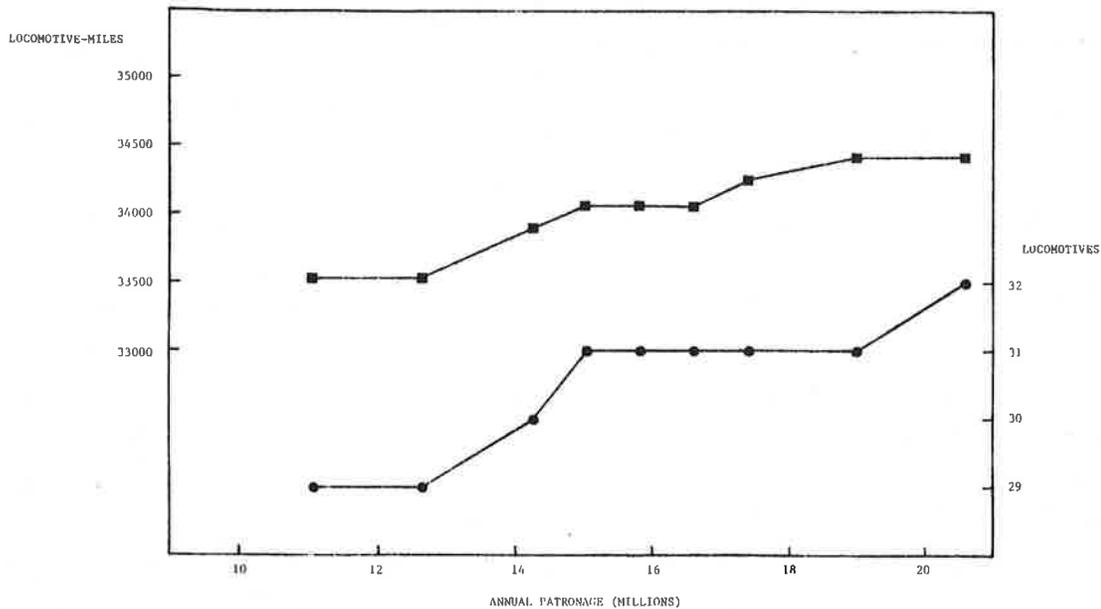
Figure 4-3 shows cars and car-miles plotted against total annual patronage for the case in which capital cost predominates. These slightly convex curves are fairly close to lines through the origin, especially within a limited range (say, 20% around the base data). Hence as a rule of thumb one may say that both the minimum fleet size, and the minimum number of car-miles that must be run with minimal fleet, are roughly proportional to total patronage:

$$\begin{aligned} \min Z_{\text{CAR}} &\approx .0000103 \cdot (\text{total annual patronage}) \\ \min Z_{\text{MILE}} | Z_{\text{CAR}} &\approx .0086 \cdot (\text{total annual patronage}) \end{aligned}$$

(In fact, both cars and car-miles do approach proportionality to patronage as the latter goes to infinity. This is because at very high demands the problem is virtually continuous, so that any increase in total demand can be met by just increasing the size of each train in the same proportion, with rounding in negligible amounts. At fairly small demand, on the other hand, the integrality of the problem comes into play. A relatively large amount of excess capacity is run simply because demands are rounded up to the next integer, and hence the actual curve for cars or car-miles runs somewhat above the line of proportionality - see small graph in Figure 4-3.

Many more sophisticated sensitivity analyses are conceivable if one allows patronage between different station-pairs to vary at different rates. For example, one might use annual patronages computed under different assumptions; or one might apply different cumulative probability distributions to one set of annual patronages.

Figure 4-4. Minimum locomotives (right scale) and locomotive-miles (left scale) as a function of annual patronage. In all cases, both objectives can be minimized by a single solution.



#### §4.7 Locomotive requirements

The upper-limits rules for the base run (§3.7) insure that the number of 14-car sections that must be run to meet each train's demand is fixed: if demand is 14 cars or less, one section is run; if demand is greater than 14 but not more than 28, two sections are run; and so forth. Hence, assuming one locomotive per section, one can tell exactly how many locomotives will be required for each train in the schedule, in any feasible solution. The analysis of §1.7 is thus applicable: locomotive demands can be substituted for car demands to determine the number of locomotives required and how far they must travel.

Only 5 trains in the base run required two sections (and hence two locomotives); the remainder all required one. One-section trains were given an upper limit of two locomotives, and two-section trains an upper limit of three (for up to 21 cars) or four (for 22-28 cars). Sets of limits were computed and filed by the same demand program used for modeling cars (§4.1).

Solving the model with the techniques applied previously to car demands, it was determined that a single solution minimized both the number of locomotives required (31) and the number of locomotive-miles run (34074). Only 4 sections, all southbound, had to be run with an extra locomotive.

Sensitivity analyses analogous to those run for car demands (§4.6) were also applied to locomotives. Results are plotted in Figure 4-4. The case at 70% of base demand requires only one locomotive for every scheduled train; hence 29 locomotives is an absolute minimum for the base schedule.

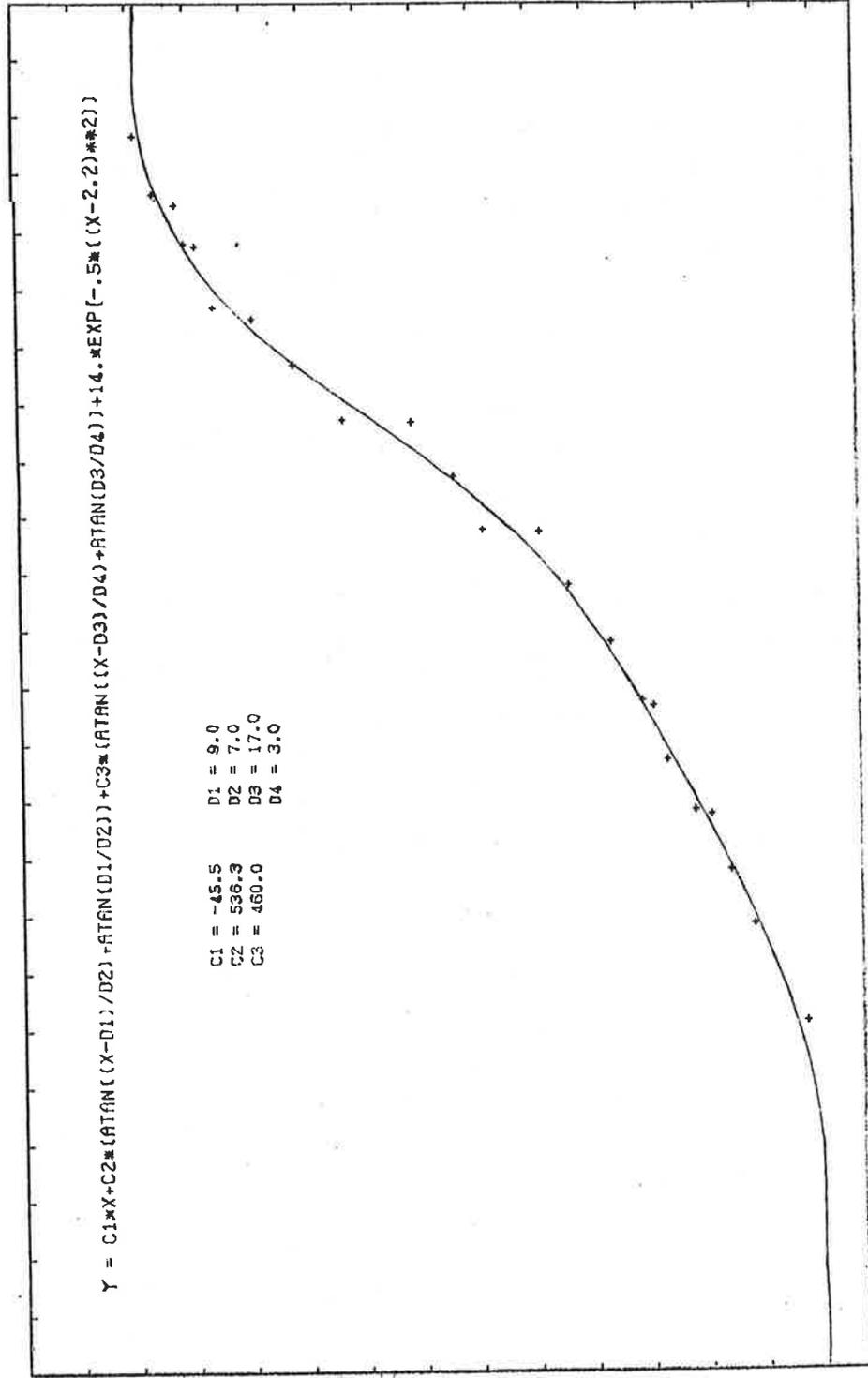
#### §4.8 Printing solution data

A set of DATAMAT programs was written to print data of interest derived from the solution at any basis. Information available from these programs includes:

- Load factors
- Train sizes
- Distribution of train sizes
- Total cars in storage at each city and time

Further information is provided in Volume 2 of this report [4].

Figure A-3. A better fit to the raw data. The general functional form here is considerably more involved, however.



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