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Index coding, a coding formulation traditionally analyzed in the theoretical computer science and information theory communities, has received considerable attention in recent years due to its value in wireless communications and networking problems. In particular, there is a now well understood correspondence between interference alignment and index coding problems, and a deep relationship between unsolved network coding problems and index coding. This renewed attention on index coding has resulted in a growing body of literature on new bounds, coding strategies and applications of index coding. We developed index-coded retransmission schemes to increase spectral efficiency of retransmission subchannels. We also defined index coding gain as a performance measure to evaluate the schemes. The numerical result shows that even with a suboptimal rank minimization algorithm, the average achievable gain is significant.
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Abstract—This paper presents a novel retransmission strategy for communication networks based on index coding. The particular example of OFDMA downlink networks is taken to illustrate its efficacy. The benefits and challenges in using index coding as a transmission strategy are highlighted. The paper concludes with characterization of expected index coding gain in terms of channel parameters.

I. INTRODUCTION

Index coding, a coding formulation traditionally analyzed in the theoretical computer science and information theory communities [1]–[7], has received considerable attention in recent years due to its value in wireless communications and networking problems. In particular, there is a now well-understood correspondence between interference alignment and index coding problems [5], [9], and a deep relationship between unsolved network coding problems and index coding [8]. This renewed attention on index coding has resulted in a growing body of literature on new bounds, coding strategies and applications of index coding.

One particular application of index coding is in the wireless downlink domain, as evidenced in literature [11]–[13]. Indeed, several wireless networking problems, downlink networks in particular, can be understood to be similar to index coding problems. Although this theoretical correspondence between index coding and wireless problems is valuable, our focus in this paper takes it further: to find index coding strategies for realistic wireless networks. To this end, in this paper, we characterize the benefits of index coding in an OFDMA downlink. In particular, we characterize the benefits of index coding for retransmissions within such a downlink.

The core idea can be illustrated as follows. There are $K$ active users in an OFDMA downlink network, and each of them is scheduled on a frequency subchannel, a subset of subcarriers. Suppose, in time frame $m - 1$, three users did not successfully decode their desired messages and are to be scheduled for retransmission in time frame $m$. In OFDMA downlink, receivers can often decode the other users’ physical layer signals on the other subchannels. These can be exploited as side information to cancel the interference in subsequent retransmission.

One such scenarios is illustrated in Fig. 1: receiver 1 knows message $w_2$ for receiver 2, receiver 2 knows messages $w_1$ and $w_3$ for receiver 1 and 3, respectively, and receiver 3 knows message $w_1$ for receiver 1. Without index coding, three messages are transmitted on separate radio resource blocks (RB) in time or in frequency. In contrast, with index coding, the messages $w_1$ and $w_2$ can be combined, e.g., $w_1 + w_2$ and sent on the same RB. At receivers 1 and 2, after the combined message is decoded, the desired message can be recovered by making use of the side information: $(w_1 + w_2) - w_2 = w_1$ at receiver 1 and similarly for receiver 2.

In contrast to the original index coding formulation, its application to wireless networks has to deal with decoding error probabilities due to channel variations such as fading. Given the basic principle and new requirement, we make this mathematically rigorous in the next few sections of the paper.

II. SYSTEM MODEL

We consider OFDMA downlink transmission where the base station transmitter sends signals to $K$ receivers on $L$ parallel (frequency) subchannels. The transmitter sends $x = [x_1 \ x_2 \ \cdots \ x_L]^T$. At receiver $k$, the received signal is

$$y_k = H_k x + z_k$$

where $H_k = \text{diag}(h_{k,1}, h_{k,2}, \ldots, h_{k,L})$ is a diagonal matrix, and $z_k = [z_{k,1} \ z_{k,2} \ \cdots \ z_{k,L}]^T$. Here, $y_{k,l}, h_{k,l}, z_{k,l}$ and $x_l$ are complex numbers, and $z_{k,l}$ is i.i.d. circularly symmetric complex Gaussian, $CN(0, 1)$. The notation $\{h_{k,l}\}$ denotes the set of all $h_{k,l}$ for $k = 1, 2, \ldots, K$ and $l = 1, 2, \ldots, L$.

Each subchannel is an abstraction of aggregate OFDM subcarriers, not necessarily contiguous, to be scheduled for a receiver. For example, in LTE networks, a subchannel is equivalent to one or more resource blocks (RB), each of which consists of 12 subcarriers with 15 kHz subcarrier spacing. Due to orthogonality of subcarriers, there is no interference between subchannels. When it is necessary, we also use the notation with time slot index $t$, $y_{k,t} = H_{k,t}[t][x[t] + z_{k,t}]$ for $t \in \{1, 2, \ldots, n\}$. The symbols $\{x[t]\}_{t=1}^n$ over time on subchannel $l$ form a coding block. A time frame is $n$ time slots for a coding block of $n$ symbols. One subchannel and one time frame is a basic unit for multi-user scheduling. The transmit signal is subject to average-power constraints, i.e.,

$$\sum_{t=1}^n |x[t]|^2 = nP_l$$

in addition to the sum-power constraint:

$$\sum_{l=1}^L P_l = P.$$ 

We now describe the statistical characteristics of $H_k$.

- The magnitude $|h_{k,l}|$ is Rayleigh distributed, and the phase $\phi(h_{k,l})$ is uniformly distributed on $[0, 2\pi]$. 

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The diagonal elements of $H_k$ are independent and identically distributed (frequency-selective fading).

- The matrices $H_1, H_2, \ldots, H_K$ are independent of each other (but not necessarily identically distributed).
- The matrix $H_k$ is realized once in each time frame, i.e., $H_k[1] = H_k[2] = \ldots = H_k[n] = H_k$ (block fading).
- The realizations $H_k[1], H_k[n+1], H_k[2n+1], \ldots$ over subsequent time frames are independent and identically distributed.

Since we consider block fading model, the channel for coded block transmission can be expressed as $Y_k = H_k X + Z_k$ with $X = [x[1] \ x[2] \ \cdots \ x[n]]$, $Z_k = [z_k[1] \ z_k[2] \ \cdots \ z_k[n]]$. We also use expression $X = [x_1 x_2 \ \cdots \ x_L]^T$ where the row vector $x_l^T = [x_l[1] x_l[2] \ \cdots x_l[n]]$ is a coded block on subchannel $l$.

We assume perfect channel state information (CSI) available at the receivers, but no CSI at the transmitter. Suppose receiver $k$ with code rate $R_k$ is scheduled on subchannel $l$ with power allocation $P_l$. The subchannel $l$ at receiver $k$ is in outage if the realization of $h_{k,l}$ does not support $R_k$, i.e., $\log(1 + |h_{k,l}|^2 P_l) < R_k$. We model the subchannel $l$ at receiver $k$ as an erasure channel with erasure random variable $\varepsilon_{k,l} = \{0, 1\}$ and erasure probability

$$
\varepsilon_{k,l} = \mathbb{P}\{\varepsilon_{k,l} = 1\} = \mathbb{P}\{h_{k,l} < \sqrt{\frac{2R_k - 1}{P_l}}\}. \tag{2}
$$

For Rayleigh fading channel, $\varepsilon_{k,l} = 1 - \exp\left(-\frac{2^{\frac{R_k - 1}{P_l}} - 1}{g_{k,l}}\right)$ where $g_{k,l} = \mathbb{E}[|h_{k,l}|^2]$. Since $\{h_{k,l}\}$ are independent, so are $\{\varepsilon_{k,l}\}$.

The expected spectral efficiency for receiver $k$ on subchannel $l$ is defined as $R_k$ weighted by success probability, i.e.,

$$
\rho_{k,l} = R_k (1 - \varepsilon_{k,l}) = R_k \exp\left(-\frac{2R_k - 1}{\text{SNR}_{k,l}}\right) \tag{3}
$$

bits per channel use where $\text{SNR}_{k,l} = g_{k,l} P_l$ is determined by the power allocation $(P_1, P_2, \ldots, P_L)$.

**Lemma 1 (optimal code rate):** Given $\text{SNR}_{k,l}$, $\rho_{k,l}$ is maximized at

$$
R_k^* = \frac{W(\text{SNR}_{k,l})}{\ln(2)}
$$

where $W(\cdot)$ is the Lambert W-function.

**Proof:** By differentiating $\rho_{k,l}$ with respect to $R_k$ and setting it to zero, we get

$$
\left(1 - \frac{2^{R_k - 1}}{\text{SNR}_{k,l}}\right) \exp\left(-\frac{2R_k - 1}{\text{SNR}_{k,l}}\right) = 0. \tag{4}
$$

Here, $\exp\left(\frac{2R_k - 1}{\text{SNR}_{k,l}}\right) > 0$, so $\frac{2^{R_k - 1}}{\text{SNR}_{k,l}} = 1$. By rearranging, we get $\text{SNR}_{k,l} = R_k^* \frac{\ln(2)}{W(\text{SNR}_{k,l})}$. The solution for the equation of the form $We^W = x$ is the Lambert W-function $W(x)$, thus $R_k^* \frac{\ln(2)}{\text{SNR}_{k,l}} = W(\text{SNR}_{k,l})$.

Since $\{h_{k,l}\}_{l=1}^L$ are i.i.d., we can assume that receiver $k$ is scheduled on subchannel $l = k$ without loss of generality. The sum efficiency is defined by $\sum_{k=1}^K \rho_{k,k}$. Since the terms $\rho_{k,k}$ are not coupled, each term can be optimized over individual $R_k$ at $R_k^* = \frac{W(\text{SNR}_{k,k})}{\ln(2)}$.

### III. INDEX-CODED RETRANSMISSION

Let us define the message for receiver $k$ as a column vector of bounded integers, i.e., $w_k \in \mathbb{W}_k = \{0, 1, \ldots, p-1\}^{nR_k}$. For simplicity, $nR_k$ is assumed to be an integer. In below, we use the notation $W = [w_1 \ w_2 \ \cdots \ w_K]^T$.

**A. Retransmission scenario**

In time frame $m$, a part of receivers are in the retransmission group $\mathcal{U}_{m-1} \subseteq \{1, 2, \ldots, K\}$ if the desired messages of the receivers were erased in time frame $m-1$ and to be scheduled for retransmission in time frame $m$. A conventional approach to retransmission is to send each individual message on a separate subchannel, which requires $q = |\mathcal{U}_{m-1}|$ subchannels.

Based on index coding, our new approach is to send linear combinations of messages, which we refer to equations, on $r < q$ subchannels. Note that $q = 1$ is a trivial case, and we only need to consider $q \geq 2$ cases.

If the message for receiver $k$ on subchannel $l = k$ in time frame $m-1$ is erased, the receiver looks at the other subchannels $l \neq k$, collects the other messages as side information if they are decodable. It then sends the transmitter a feedback with the list of side information $\mathcal{A}_{k,m-1} = \{1, 2, \ldots, K\}\backslash\{k\}$. Note that the receivers only feedback the list of IDs of messages, not the message itself. $\mathcal{A}_{k,m-1}$ can be equivalently expressed in matrix form $A_k$ by listing up all the side information as row vectors with a single support, as described in the following example.

**Example 1:** Consider $\mathcal{U}_{m-1} = \{1, 2, 3\}$ with $A_{1,m-1} = \{2\}, A_{2,m-1} = \{1, 3\}, A_{3,m-1} = \{1\}$. The equivalent matrix representations for $A_{k,m-1}$ are

$$
A_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},
A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
A_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
$$

respectively. As a result,

$$
A_1 W = w_2^T, \quad A_2 W = [w_1 \ w_3]^T, \quad A_3 W = w_1^T,
$$

respectively.
Having received the feedback from the receivers, the transmitter can construct a $K \times K$ dashboard matrix $F$. In $k$-th row of $F$,
- the $(k, k)$ position is set to 1 if receiver $k$ decoded the desired message $w_k$ successfully, otherwise set to 0,
- the $(k, l \neq k)$ position is set to ($\ast$) if receiver $k$ decoded the message $w_l$ successfully, otherwise set to 0.

Example 2: An example of the feedback dashboard matrix with $K=5$ is

$$F = \begin{bmatrix} 1 & 0 & 0 & \ast & 0 \\ 0 & 0 & \ast & 0 & 0 \\ 0 & \ast & 0 & 0 & \ast \\ \ast & 0 & \ast & 1 & \ast \\ \ast & \ast & 0 & 0 & 0 \end{bmatrix}.$$  

By looking at the diagonal, we know that the receivers $k=1,4$ successfully decoded their desired messages, thus $U_{m-1} = \{2,3,5\}$ and $q = |U_{m-1}| = 3$.

Note that $k$-th row of $F$ is the feedback from receiver $k$. The $l$-th column shows the erasure pattern on subchannel $l$ across the receivers. We can get the submatrix $F_{lU_{m-1}}$ for the retransmission group in the following way: remove the $k$-th row and $k$-th column if $(k, k)$ element is set to 1, otherwise set to 0.

Example 3: By removing first and fourth rows and columns from $F$ in the previous example, we get

$$F_{lU_{m-1}} = \begin{bmatrix} 0 & \ast & 0 \\ \ast & \ast & \ast \\ \ast & \ast & 0 \end{bmatrix}.$$  

Fig. 3 shows examples of randomly generated $F_{lU_{m-1}}$ when $\{\epsilon_{k,l}\}$ are i.i.d., thus $\epsilon_{k,l} = \epsilon$ for all $k$ and $l$. In this particular case, the average dimension of $F_{lU_{m-1}}$ is $E[q] = \epsilon K$. As indicated by the off-diagonal elements of $F_{lU_{m-1}}$ in the figure, for reasonable values of $\epsilon$, the receivers in the retransmission group acquire a lot of side information. That is the main enabler of our retransmission strategy based on index coding.

B. Index coding

In the sequel, we drop the time frame index $m-1$ for simplicity. The side information graph $\mathcal{G}$ for $\mathcal{U} = \{1,2,\ldots,q\}$ in a compact form [6] is defined by

$$\mathcal{G} = \{(1|A_1), (2|A_2), \ldots, (q|A_q)\}.$$  

Note that describing $\mathcal{G}$ is equivalent to describing $F_{lU}$ and $U$.

Example 4: Consider $\mathcal{U} = \{1,2,3\}$ with $A_1 = \{2\}$, $A_2 = \{1,3\}$, $A_3 = \{1\}$. The side information graph $\mathcal{G} = \{(1|2), (2|1,3), (3|1)\}$ is a directed graph described in Fig. 2.

Without index coding, we need $q = |\mathcal{U}|$ subchannels to retransmit $q$ messages. The role of index coding is to reduce the number of required subchannels for retransmission from $q$ to $r$. A rank minimization algorithm takes a $q \times q$ matrix $A_\mathcal{G} = 1 + F_{lU}$ as an input and outputs a $q \times q$ matrix $B_\mathcal{G}$ with ($\ast$) positions filled in. In this paper, we consider index codes over real numbers.

Example 5: A rank minimization algorithm takes input $A_\mathcal{G} = 1 + F_{lU}$ with the $F_{lU}$ in the previous example and outputs

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$  

with $r = \min\text{rank}(A_\mathcal{G}) = \text{rank}(B) = 2$.

There are several ways of finding two linearly independent equations from $B$. In below, we explain an SVD-based method [12]. We can find the singular value decomposition (SVD), $B = U \Sigma V^T$. There are $r$ nonzero singular values, i.e., $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r, 0, 0, \ldots, 0)$ where we assume $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ without loss of generality. The matrix $DV^T$ is $q \times q$, but the bottom $q-r$ rows are all zero.

Example 6: From $B$ above, we get

$$DV^T = \begin{bmatrix} -1.4142 & 0.7071 & 0.7071 \\ 0 & -1.2247 & 1.2247 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Then, we can find an $r \times q$ index coding matrix $G$ with rank-$r$ by taking the $r$ non-zero rows, which we denote by $G = [DV^T]_{1:r}$.

C. Sending equations and solving at the receivers

Given the messages $\{w_1, w_2, \ldots, w_q\}$, we make $r$ linear combinations of the messages:

$$S = \begin{bmatrix} s_1^T \\ s_2^T \\ \vdots \\ s_r^T \end{bmatrix} = G \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_q^T \end{bmatrix} = GW,$$

where we call $s_r^T$ an equation. The transmit signal for retransmission is

$$X = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_r^T \end{bmatrix} = \begin{bmatrix} f_1(s_1)^T \\ f_2(s_2)^T \\ \vdots \\ f_r(s_r)^T \end{bmatrix} = f(S) = f(GW),$$

where $f_i(\cdot)$ are the encoding functions for error correction coding. Now, the transmitter sends both $X$ and the coefficient...
matrix $B$. Note that the overhead of sending $B$ is negligible if the messages $w_k$ are large enough in data size. We need $L = K + r$ subchannels: $r$ subchannels dedicated for retransmission, and $K$ subchannels for new information transmission.

Receiver $k$ in $U$ first decodes all the equations $\{s_i\}$ that involve its desired message $w_k$, and then solves the equations for the message. The equation $s_i$ can be recovered if $f(s_i)$ is decoded successfully without error. Then, $D V^T W$ can be formed by adding all-zero rows, i.e.,

$$SS = GWW = GVW = D V^T W.$$  

Having received $B$ from the transmitter, the receiver can find $U$ by performing SVD, and then calculate

$$U(DV^T W) = (UDV^T)W = BW.$$  

At receiver $k$, the desired message $w_k$ can be recovered by combining $BW|_{k}$ and $A_{k}W$ where $[.]_{k}$ denotes the $k$-th row of the matrix.

**Example 7:** For our example at hand,

- at receiver 1,

$$[BW]_1 + A_1W = (w_1 - w_2^T) + w_2^T = w_1^T,$$

- at receiver 2,

$$[BW]_2 + A_2W = (w_2 - w_3^T) + w_3^T = w_2^T,$$

- at receiver 3,

$$[BW]_3 + A_3W = (-w_1^T + w_3^T) + w_1^T = w_3^T.$$

**D. Performance criteria**

Erasure probability of an equation $s_i^T$ on subchannel $l$ at receiver $k$ is given by

$$\epsilon_{k,l} = \mathbb{P}\left\{ |h_{k,l}| < \sqrt{\frac{\sum_{j \in N_l} R_j - 1}{P_l}} \right\}$$

$$= 1 - \exp\left( -\frac{\sum_{j \in N_l} R_j - 1}{g_{k,l} P_l} \right)$$

where $N_l$ denotes the set of messages involved in the equation $s_i^T$ on subchannel $l$. Recovery of a retransmit message $w_k$ is successful if all the equations $\{s_i\}_{i \in M_k}$ that involve the message are successfully decoded. Since $\{\epsilon_{k,l}\}$ are independent, the expected spectral efficiency for retransmission of message $w_k$ is given by

$$R_k \prod_{l \in M_k} (1 - \epsilon_{k,l}).$$

As $q$ messages are delivered in $r$ subchannels the expected spectral efficiency for retransmission subchannels is given by

$$\rho_{\text{RTX}} = \frac{1}{r} \sum_{k=1}^{q} R_k \prod_{l \in M_k} (1 - \epsilon_{k,l}).$$

Without index coding, we have the baseline spectral efficiency of $q$ subchannels given by

$$\rho_0 = \frac{1}{q} \sum_{k=1}^{q} R_k (1 - \epsilon_{k,l}).$$

The index coding gain is defined by

$$\eta = \frac{\rho_{\text{RTX}}}{\rho_0}.$$

To get some intuition, let us consider a special case with $R_k = R$ and $\text{SNR}_{k,l} = q_{k,l} P_l = \text{SNR}$ for all pairs of $k$ and $l$. Note that given $q_{k,l} P_l = \mathbb{E}[|h_{k,l}|^2]$, we can choose the power allocation $P_l = \text{SNR}_{k,l} q_{k,l}$ to satisfy $\text{SNR}_{k,l} = \text{SNR}$. The constant SNR can be determined to meet the transmit power constraint $\sum_{l=1}^{L} P_l = \sum_{l=1}^{L} \frac{\text{SNR}}{M_{k,l}} = P$, thus $\text{SNR} = \frac{P}{\sum_{l=1}^{L} q_{k,l}}$. Since $R_k = R$ and $\text{SNR}_{k,l} = \text{SNR}$, it follows that $\epsilon_{k,l} = \epsilon$ with

$$\epsilon = 1 - \exp\left( -\frac{2R - 1}{\text{SNR}} \right).$$

For this special case, we have the following result.

**Theorem 1 (lower bound on index coding gain):** For the case of $R_k = R$ and $\text{SNR}_{k,l} = \text{SNR}$, the index coding gain $\eta$ is lower bounded by

$$\eta \geq \eta_{\text{LB}} = \frac{q}{r} \exp\left( -\frac{r - 1}{\text{SNR}} \left(2R - 1\right) \right).$$

In addition, $r \leq |r_{\text{th}}|$ is a sufficient condition on $r$ for index coding gain $\eta \geq \eta_{\text{LB}} \geq 1$ where $r_{\text{th}}$ is the solution for the equation $\frac{q}{r} \left(1 - \epsilon\right)^{r-1} = 1$.

**Proof:** Since $M_k \subseteq \{1, 2, \ldots, r\}$ and $|M_k| \leq r$, we get the lower bound

$$R_k \prod_{l \in M_k} (1 - \epsilon_{k,l}) \geq R_k \prod_{l=1}^{r} (1 - \epsilon_{k,l})$$

for any $k \in U$. Thus,

$$\rho_{\text{RTX}} \geq \rho_{\text{LB}} = \frac{1}{r} \sum_{k=1}^{q} R_k \prod_{l=1}^{r} (1 - \epsilon_{k,l}),$$
and \( \eta \geq \eta_{LB} \) where
\[
\eta_{LB} = \frac{\rho_{LB}}{\rho_0} \tag{7}
\]
\[
\frac{q}{r} \cdot \frac{\sum_{k=1}^{r} R_k \prod_{l=1}^{q} (1 - \epsilon_{k,l})}{\sum_{k=1}^{r} R_k (1 - \epsilon_{k,l})}. \tag{8}
\]
Since \( R_k = R \) and SNR\(_{k,l} = \text{SNR} \), it follows that \( \epsilon_{i,k} = \epsilon \) for all pairs of \( k \) and \( l \). As a result,
\[
\eta_{LB} = q/(1 - \epsilon)^{r-1}.
\]

Since \( 1 - \epsilon = \exp\left(-\frac{2q}{\text{SNR}}\right) \), the result follows. The sufficient condition for \( \eta \geq 1 \) follows immediately.

Remark 1: If \( r = q \), then \( \eta_{LB} = (1 - \epsilon)^{q-1} < 1 \) for \( \epsilon > 0 \), we should send an individual message on each subchannel rather than an equation. In contrast, if \( r = 1 \), then \( \eta_{LB} = q \), which is the maximum possible gain by having side information.

Fig. 4 shows the plot of \( \eta_{LB} = q/(1 - \epsilon)^{r-1} \) over \( r \) with \( \epsilon = 0.1 \) and \( q = 10 \). As the plot shows, in this case, \( r \leq 5 \) is required to achieve \( \eta_{LB} \geq 1 \).

E. Rank minimization for index coding

Given the side info matrix \( A_G \), the rank minimization is the optimization problem,
\[
\min_{B \in \mathbb{R}^{s \times s}} \{ \text{rank}(B) : [B]_{i,j} = [A_G]_{i,j} \text{ if } [A_G]_{i,j} \neq * \}
\]
where \([B]_{i,j}\) denotes the element in \((i,j)\) position of \( B \). This is not a convex optimization problem since the rank of a matrix is not a convex function. In matrix completion literature, the convex relaxation with nuclear norm is often used [10], i.e.,
\[
\min_{B \in \mathbb{R}^{s \times s}} \{ \|B\|_{\text{nuc}} : [B]_{i,j} = [A_G]_{i,j} \text{ if } [A_G]_{i,j} \neq * \}
\]
where the nuclear norm \( \|B\|_{\text{nuc}} \) is the sum of singular values of \( B \). However, as pointed out in [13], the nuclear norm-based methods do not work well for index coding problems. For the numerical results in this paper, the alternating projections algorithm in [13] is used. Let us define two sets:
\[
B_r = \{B : \text{rank}(B) \leq r\},
\]
\[
B_G = \{B : [B]_{i,j} = [A_G]_{i,j} \text{ if } [A_G]_{i,j} \neq *\},
\]
and the projection operation is defined by
\[
\text{Proj}_B(B') = \arg\min_{B \in B''} \{ \|B - B'\|_F \}
\]
where \( \| \cdot \|_F \) is the frobenious norm. The algorithm performs
1) For \( i = 0 \), set initial values \( B(0) \) and \( r(0) \).
2) For \( i \geq 1 \), find \( B' = \text{Proj}_{B_{G_{i-1}}}(B^{(i-1)}) \) and then find \( B^{(i)} = \text{Proj}_{B_G}(B') \).
3) If \( \text{rank}(B^{(i)}) \leq r^{(i-1)} \), set \( r^{(i)} = r^{(i-1)} - 1 \), go to 2).
Otherwise, return \( \text{rank}(B^{(i)}) \) and \( r^{(i-1)} \), and terminate.
Given $F$ of $R$ for $100$ realizations of $F$ show either $\eta$ observed no case of negative $\epsilon$. $\eta$ how much achievable gain is significant.

With a suboptimal rank minimization algorithm, the average evaluate the schemes. The numerical result shows that even to arbitrarily correlated subchannels based on the Markov model: $h_{k,l} = u_{k,l,1}$, and $h_{k,l} = \beta h_{k,l-1} + \sqrt{1-\beta^2}u_{k,l}$ for $l \geq 2$ where $\{u_{k,l}\}_{l=1}^L$ are i.i.d.

**IV. Numerical Results**

We begin by explaining experiment setup. We assume that $R_k = R$, and by some choice of $P_l$ given $g_{k,l}$, $\epsilon_{k,l} = \epsilon$ for all $k$ and $l$. In other words, $\{\epsilon_{k,l}\}$ are i.i.d. Given $\epsilon$, we randomly generate $F$ with $K = 100$. For each realization of $F$, we do the following steps:

- find $U$ and $A_G$.
- find $r = \min\text{Rank}(A_G)$ by running alternating projections algorithm,
- calculate $\eta_{LB} = \frac{2}{r}(1-\epsilon)^{r-1}$.

Given the $\eta_{LB}$ versus $r$ curve in Fig. 4, we are intested in how much $\eta_{LB}$ is achievable by a particular rank minimization algorithm. We generate many realizations of random $F$ with $\epsilon = 0.1$ and choose 1000 of them with $q = |U| = 10$. Given $F_{U\epsilon}$, equivalently $A_G$, we run the alternating projections algorithm to find $r$. Fig. 5. (a) shows the empirical distribution of $r$ and the corresponding $\eta_{LB}$. In this experiment, we observed no case of negative $\eta_{LB}$. Most of the realizations of $F$ show either $\eta_{LB} = 4.5$ or 2.7. Similarly, we also generate 100 realizations of $F$ with $\epsilon = 0.2$ and find the ones with $q = 20$. The result is summarized in Fig. 5. (b).

Fig. 6 and Fig. 7 show the empirical averages of $q$ and $r$, and $\eta_{LB}$ for 100 randomly generated $F$. For $\epsilon \leq 0.1$, the impact of the factor $q$ increasing in $\epsilon$ is dominant, so $\eta_{LB}$ is increasing in $\epsilon$. In constrast, for $\epsilon > 0.1$, the impact of the factor $\frac{2}{r}(1-\epsilon)^{r-1}$ is stronger, so $\eta_{LB}$ is decreasing in $\epsilon$.

**V. Conclusion**

We developed index-coded retransmission schemes to increase spectral efficiency of retransmission subchannels. We also defined index coding gain as a performance measure to evaluate the schemes. The numerical result shows that even with a suboptimal rank minimization algorithm, the average achievable gain is significant.

For future work, we consider the extension to larger retransmission window with $U_{m-1}, U_{m-2}, \ldots, U_{m-r}$. Another interesting research direction is to formulate radio resource optimization problems to jointly optimize the system parameters. In this paper, we assumed that $h_{k,l}$ across subchannels $l = 1, 2, \ldots, L$ are independent. The results may be extended to arbitrarily correlated subchannels based on the Markov model: $h_{k,l} = u_{k,l,1}$, and $h_{k,l} = \beta h_{k,l-1} + \sqrt{1-\beta^2}u_{k,l}$ for $l \geq 2$ where $\{u_{k,l}\}_{l=1}^L$ are i.i.d.

**REFERENCES**